ENGINEERING & TECHNOLOGY EXAMINATIONS, DECEMBER – 2005

MATHEMATICS

SEMESTER – 1

Time : 3 Hours] [Full Marks : 70

The questions are of equal value.
The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Note : i) Question No. 1 is compulsory.

ii) Answer any *six* full questions from the remaining.

1. Answer any *five* of the following questions : $5 \times 2 = 10$

i) Show that the sequence $\{ U_n \}_{n \in N}$, where $U_n = 2(-1)^n$ does not converge.

ii) Use L'Hospital's rule to evaluate $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$.

iii) If $u = \log(\tan x + \tan y)$, prove that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} = 2$.

iv) Show that Lagrange's Mean Value Theorem is not applicable to the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases} \quad \text{in } [-1, 1].$$

v) Evaluate the line integral $\int_C (x^2\, dx + xy\, dy)$, where C is the line segment joining $(1, 0)$ and $(0, 1)$.

vi) If α, β, γ are the angles which a line makes with the coordinate axes, prove that $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$.

vii) If $|\vec{\alpha}| = 3$ and $|\vec{\beta}| = 4$, then find the values of the scalar c for which the vectors $\vec{\alpha} + c\vec{\beta}$ and $\vec{\alpha} - c\vec{\beta}$ will be perpendicular to one another.

viii) Find the unit vector normal to the surface $x^2 + y - z = 1$ at the point $(1, 0, 0)$.

[Turn over

2. a) Test the convergence of any *two* of the following series : $2 \times 3 = 6$

i) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \ldots\ldots \infty$

ii) $\sin\left(\frac{1}{1^{3/2}}\right) + \sin\left(\frac{1}{2^{3/2}}\right) + \sin\left(\frac{1}{3^{3/2}}\right) + \sin\left(\frac{1}{4^{3/2}}\right) + \ldots\ldots \infty$

iii) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots\ldots \infty$.

b) State D' Alembert's Ratio test for infinite series of positive terms. Discuss the convergence of the series $\sum_{n=1}^{\infty} n^4 e^{-n^2}$. $1 + 3$

3. a) If $y = \tan^{-1} x$, then prove that

$$\left(1 + x^2\right) y_{n+1} + 2\, nxy_n + n(n-1)\, y_{n-1} = 0.$$

Also find $y_n(0)$. $3 + 3$

b) Using Mean Value Theorem, prove that $\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$. 4

4. a) Find the value of $\lim_{n \to \infty} \left\{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)\ldots\ldots\left(1 + \frac{n}{n}\right)\right\}^{1/n}$. 4

b) If $I_n = \int \frac{\cos n\theta}{\cos\theta}\, d\theta$, show that $(n-1)\left(I_n + I_{n-2}\right) = 2\sin(n-1)\theta$.

Hence evaluate $\int \left(4\cos^2\theta - 3\right) d\theta$. $4 + 2$

5. a) Find the whole length of the loop of the curve $9y^2 = (x-2)(x-5)^2$. 4

b) Find the surface area generated by revolving the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis of the first quadrant. 4

c) If $f(x, y, z, w) = 0$, prove that $\frac{\partial x}{\partial y} \times \frac{\partial y}{\partial z} \times \frac{\partial z}{\partial w} \times \frac{\partial w}{\partial x} = 1$. 2

6. a) Find the extrema of the function $x^3 + y^3 - 3x - 12y + 20$. 4

b) If $f\left(v^2 - x^2,\ v^2 - y^2,\ v^2 - z^2\right) = 0$, where v is a function of x, y, z, show that $\frac{1}{x}\frac{\partial v}{\partial u} + \frac{1}{y}\frac{\partial v}{\partial y} + \frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}$. 3

c) Evaluate $\iint_R \sqrt{4x^2 - y^2}\, dx\, dy$, where R is the triangular region bounded by the lines $y = 0$, $x = 1$ and $y = x$. 3

7. a) Find the volume V of a solid bounded by $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$. 5

b) Find the moment of inertia of the solid bounded in the first octant by the coordinate planes and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, $(a > 0,\ b > 0,\ c > 0)$, (ρ is the constant density of the solid) about the x-axis. 5

8. a) A variable plane passes through a fixed point (a, b, c) and meets the coordinate axes at A, B, C. Show that the locus of the point of intersection of the planes through A, B, C and parallel to the coordinate planes is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$. 5

b) A straight line with direction ratios 2, 7, − 5 is drawn to intersect the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ and $\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$. Find the coordinates of the points of intersection and length intercepted on it. 5

9. a) Given two vectors $\vec{\alpha} = 3\hat{i} - \hat{j} + 0\hat{k}$ and $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$, express $\vec{\beta}$ in the form $\vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$. 3

b) Given three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, prove that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}.$$ 4

c) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that grad $f(r) \times \vec{r} = \theta$, where θ is the null vector. 3

10. a) Prove that $curl\,(\text{grad}\,(f)) = \theta$, where θ is the null vector. 2

b) Verify Green's theorem in the plane for $\oint_{\Gamma} (x^2\,dx + xy\,dy)$, where Γ is the square in the xy-plane given by $x = 0,\ y = 0,\ x = a,\ y = a\ (a > 0)$ described in the positive sense. 5

c) Evaluate by Divergence theorem $\iint_S \{x^2\,dydz + y^2\,dzdu + 2z(xy - x - y)\,dxdy\}$, where S is the surface of the cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$. 3

11. a) Show that

$$\iiint \frac{dx\,dy\,dz}{(x + y + z + 1)^3} = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]$$

integration being taken over the volume bounded by the co-ordinate planes and the plane $x + y + z = 1$. 5

b) Find the Moment of Inertia of a thin uniform lamina in the form of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis. 5