PERMUTATIONS & COMBINATIONS

Fundamental principle of Counting:

We have two principles of counting the "**Sum rule"** and the "**Product rule".**

Event A can happen in = m ways.

Event B can happen in = n ways.

Then, the chances of either (A **or** B) happening are given by $\xrightarrow[\text{can occur in}]{}$ m + n ways **(Sum Rule)**

The chances of both (A **and** B) happening simultaneously are given by $\xrightarrow[\text{can occur in}]{}$ m × n ways **(Product Rule)**

Concept of ^nP_r and ^nC_r:

$^nP_r \rightarrow$ Number of arrangements of n different objects taken 'r' at a time

$^nC_r \rightarrow$ Number of selections of n different objects taken 'r' at a time.

Formulae of Permutation:

1. Permutations of n different things taken 'r' at a time is denoted by ^nP_r and is given by

$$\mathbf{{}^nP_r = \frac{n!}{(n-r)!}}$$

2. The total number of arrangements of n things taken 'r' at a time, in which a particular thing always occurs = $\mathbf{{}^{n-1}P_{r-1}}$.

   E.g. *The number of ways in a basketball game in which 5 out of 8 selected players can play at different positions, such that the captain always plays at the centre position* = $^{8-1}P_{5-1} = {}^7P_4 = 210$.

3. The total number of permutations of n different things taken 'r' at a time in which a particular thing never occurs = $^{n-1}P_r$.

   E.g. *The number of ways in which we can form a 4 letter word from the letters of the word COMBINE such that the word never contains B* = $^{7-1}P_4 = {}^6P_4 = 30$

4. The number of arrangements when things are not all different such as arrangement of n things, when p of them are of one kind, q of another kind, r is still of another kind and so on, the total number of permutations is given by $\frac{n!}{(p!\ q!\ r!.......)}$.

   E.g. *The total arrangements of the letters of the word "M A T H E M A T I C S" in which M, A and T are repeated twice respectively* **=** $\frac{11!}{2!2!2!}$

5. The number of permutations of n different things taking 'r' at a time when each thing may be repeated any number of times in any permutations is given by (n × n × n × n × n……………..r times) i.e. $\mathbf{n^r}$ ways.

E.g. *The total numbers of ways in which 7 balls can be distributed amongst 9 persons (when any man can get any number of balls)* = $\mathbf{9^7}$ *ways.*

Circular Permutations:

In linear permutation, we fill first place by n ways and next in (n – 1) ways and so on, but in circular arrangement we don't have any first place. So fix any object as a first place and arrange the rest (n – 1) objects around it. Hence, we have to arrange 1 less than the total number of things.

i. Number of circular permutations of n things all taken at a time = (n – 1)!

ii. Number of circular permutations of n different things taking 'r' at a time $= \frac{{}^{n}P_{r}}{r}$.

iii. If there is no difference between clockwise and anticlockwise arrangements, the total number of circular permutations of n things taking all at a time is $\frac{(n-1)!}{2}$ & the total number of circular permutations n when taking 'r' at a time all will be $\frac{{}^{n}P_{r}}{2r}$.

Formulae of Combination:

1. Number of combinations of n dissimilar things taken 'r' at a time is denoted by ${}^{n}C_{r}$ & is given by

$${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$$

2. Number of combinations of n different things taken 'r' at a time in which 'p' particular things will always occur is ${}^{n-p}C_{r-p}$

E.g. *The number of ways a basketball team of 5 players chosen from 8 players, so that the captain be included in the team* = ${}^{8-1}C_{5-1} = {}^{7}C_{4} = 35$

3. Number of combinations of n dissimilar things taken 'r' at a time in which 'p' particular things will never occur is ${}^{n-p}C_{r}$

E.g. *The number of ways a basketball team of 5 players chosen out of 10 players, such that the player named Saurav should not be included in the team* = ${}^{10-1}C_{5} = {}^{9}C_{5} = 126$.

4. The number of ways in which (m + n) things can be divided into two groups containing m & n things respectively ${}^{(m+n)}C_{n} = \frac{(m+n)!}{m!\,n!} = {}^{(m+n)}C_{m}$.

5. If 2m things are to be divided into two groups, each containing m things, the number of ways = $\frac{(2m)!}{[2(m!)^2]}$.

6. The number of ways to divide n things into different groups, one containing p things, another q things & so on is equal to $\frac{(p+q+r+.....)!}{p!.q!.r!......}$ Where {n = p + q + r +…}

Distribution of Identical Objects:

The total number of ways of dividing n identical items among r persons, each of whom can receive 0, 1, 2, or more items (≤ n) is $^{n+r-1}C_{r-1}$.

OR

The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is $^{n+r-1}C_{r-1}$.

Example:

How many non – negative integral solutions are possible for the given equation?

$$x + y + z = 16$$

Hint: Here, if we look out to the problem we will find 16 objects have to be distributed among 3 different persons (i.e. x, y, z).

Hence n = 16, r = 3 and total number of non – negative solutions = $^{16+3-1}C_{3-1} = {}^{18}C_2 = 153$.

Some important Results:

i. Number of lines with n points = $^nC_2$.

∵ For making a line exactly two points are required. So the number of ways in which we can choose two points out of n point is $^nC_2$.

Combination is used here because a line from A to B is the same as from B to A. So AB & BA are the same.

(i) n lines can intersect at a maximum of $^nC_2$ points.

(ii) Number of triangles with n points = $^nC_3$.

(iii) Number of diagonals in n sided polygon = $^nC_2 - n$

ii. The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of the groups is *not important*, is

$$\left(\frac{(mn)!}{(n!)^m}\right)\frac{1}{m!}$$

iii. The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of the groups is *important*, is

$$\left(\frac{(mn)!}{(n!)^m}\right)\frac{1}{m!}\times \mathbf{m!} = \frac{(mn)!}{(n!)^m}$$

iv. Total number of rectangles formed by **n** horizontal and **m** vertical lines in a plane = ${}^{m}C_2 \times {}^{n}C_2$.

v. The total number of ways of dividing n identical items among r persons, each one of whom, can receive 0, 1, 2, or more items (≤ n) is $\mathbf{{}^{n+r-1}C_{r-1}}$.

OR

The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is $\mathbf{{}^{n+r-1}C_{r-1}}$.

PROBABILITY

If there are n-elementary events associated with a random experiment and m of them are favorable to an event A, the probability of A happening is denoted by P (A) and is defined as the ratio m/n.

Probability of an event occurring = $\dfrac{\text{Number of favourable outcomes}}{\text{Number of all possible outcomes}}$

Thus, P (A) = $\dfrac{m}{n}$.

Probability always lies between 0 and 1

a.** Probability of a **sure event** is **1.

b.** Probability for an **impossible event** is **0.

Clearly, $0 \le m \le n$, therefore $0 \le \dfrac{m}{n} \le 1$, so that $0 \le P(A) \le 1$

Since the number of cases in which the event A will not happen is n - m, therefore, if $\overline{A}$ denotes not happening of A, then the probability $P(\overline{A})$ of not happening of A is given by

$$P(\overline{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A) \quad \text{or} \quad P(A) + P(\overline{A}) = 1$$

<u>Odd in favour and odd against:</u>

If **m** is the favourable chances of an event and **n** be the total chances of the event,

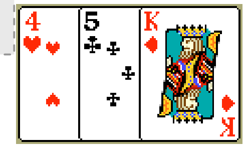
the **odds in favour** of occurrence of the event A are defined by **m : (n – m)** i.e., P (A): $P(\overline{A})$ and the **odds against** the occurrence of A are defined by **(n – m) : m,** i.e., $P(\overline{A})$: P (A).

<u>Mutually exclusive and inclusive events:</u>

Two or more events are said to be **mutually exclusive if** these events cannot occur simultaneously. Two or more events are said to be **compatible,** if they can occur simultaneously.

Two events (A and B) are mutually exclusive, if the intersection of two events is null or they have no common element i.e. $A \cap B = \phi$. And are mutually inclusive, if they have atleast one of the elements in common i.e. $A \cap B \neq \phi$.

E.g.



In drawing a card from a deck of 52 cards:

A: The event that it is a red card.

B: The event that it is a black card.

C: The event that it is a king.

In the above case events A and B are mutually exclusive but the events B and C are not mutually exclusive or disjoint since they may have common outcomes.

More precisely we can use the following formula for these two types of events.

1. If E and F two mutually exclusive events, the probability that either event E or event F will occur in a single trial is given by:

   P(E or F) or P (E $\cup$ B) = P(E) + P(F)

2. If the events are not mutually exclusive,

   P(E or **F) = P(E) + P(F) – P(E and F together)**

   or

   P(E $\cup$ F) = P(E) + P(F) – P(E $\cap$F)

   Note: P (neither E nor F) = 1 – P(E or F).

Independent Events:

When two events, A and B are independent, the probability of both occurring is:

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$$

Example:

A coin is tossed and a single 6-sided dice is rolled. Find the probability of getting a head on the coin and a 3 on the dice.

Binomial Distribution:

If n trials are performed under the same condition and the probability of success in each trial is p, and q = 1 – p then the probability of exactly r successes in n trials is:

$$\boxed{P(r) = {}^{n}C_{r}\, p^{r}\, q^{n-r}}$$

Example:

A dice is tossed 5 times. What is the probability that the number 5 shows up exactly thrice, on the dice?