INSTITUTE OF ACTUARIES OF INDIA

EXAMINATIONS

11th November 2011

Subject CT4 – Models

Time allowed: Three Hours (10.00 – 13.00 Hrs)

Total Marks: 100

INSTRUCTIONS TO THE CANDIDATES

1. ***Please read the instructions on the front page of answer booklet and instructions to examinees sent along with hall ticket carefully and follow without exception***
2. ***Mark allocations are shown in brackets.***
3. ***Attempt all questions, beginning your answer to each question on a separate sheet. <u>However, answers to objective type questions could be written on the same sheet</u>.***
4. ***In addition to this paper you will be provided with graph paper, if required.***
5. ***Please check if you have received complete Question Paper and no page is missing. If so, kindly get new set of Question Paper from the Invigilator.***

AT THE END OF THE EXAMINATION
Please return your answer book and this question paper to the supervisor separately.

Q. 1) A. R. Rehman a musician has to compose 8 tracks for a movie and typically he works on one track each week. Every Friday, A R Rehman critically reviews the track he has just composed. There is an 80% chance that he will be happy with his work and the track will be treated as final. However, there is a 20% chance that Mr. Rehman will not be satisfied; and will dismiss the just-composed track. Once A R Rehman has finalized a track, he does not revisit it at any later date.

Let T_k denote the number of tracks that Mr. Rehman has completed by the end of the k^{th} week. Let $T_0 = 0$.

i) Explain why T_k can be modeled as a Markov Chain. (2)

ii) Calculate the probability that A R Rehman will compose all 8 tracks in:

a) exactly 8 weeks

b) exactly 12 weeks (3)

iii) Calculate the expected number of weeks that A R Rehman will take to compose all the tracks. (2.5)

iv) Calculate the expected number of weeks that Mr. Rehman will take to compose remaining tracks where:

a) He has worked for 3 weeks and has already composed 3 tracks.

b) He has worked for 10 weeks and has only composed 3 tracks. (1.5)

[9]

Q. 2) **i)** State the main advantage and disadvantage of sub-dividing data, according to characteristics such as age, sex, smoker status etc., while carrying out mortality investigations for an insurance company. (2)

ii) An insurance company sells joint-life policy under which the insurance company undertakes to pay the sum assured on the death of the first life. John (born 18 December 1963) and Angelina (born 4 June 1975) purchased a joint-life policy on 27 May 2006. John died on 1 July 2010 and the insurance company paid out the entire claim amount to Angelina a month later on 1 August 2010.

The insurance company conducts separate investigations for male and female mortality between the dates 1 January 2005 and 31 December 2010.

Determine the number of days of exposure that John and Angelina contribute at each age under the respective investigations, in the calculation of:

a) Central exposed to risk (5)

b) Initial exposed to risk (2)

Clearly state any assumptions you make.

[9]

Q. 3) A scientist working at the Large Hadron Collider (LHC) in Geneva is keenly studying the behavior of a particular type of neutrino. In particular, the scientist is interested in the rate at which this neutrino decays into other particles. The scientific community believes that understanding the rate of decay of this neutrino is essential in revealing the deepest mysteries of our universe.

The scientist intends to observe the behavior of N such neutrinos by running the LHC particle accelerator for a finite period observation. Each particle will be observed until the earlier of:

- the particle decaying into another particle; or
- the particle observation being stopped (censored) on account of certain practical limitations of the LHC; or
- the completion of the period of observation

These neutrinos do not interact with one another, and therefore the observations can be treated as independent.

The variable $\mathbf{a_i}$ indicates the time at which observation for the i^{th} particle begins. The variable $\mathbf{t_i}$ indicates the time at which the observation for the i^{th} particle ends. An indicator variable $\mathbf{D_i}$ takes the value 1 if the i^{th} particle decays at the time $\mathbf{t_i}$ or 0 if this particle does not decay.

i) If the force of decay at all times is $\boldsymbol{\mu}$; derive the maximum likelihood estimate of $\boldsymbol{\mu}$. (6)

ii) State the asymptotic distribution of the maximum likelihood estimator $\acute{\boldsymbol{\mu}}$. (2)

iii) The scientist observes 16 decays out of total waiting time of 1,000 hours. Construct an approximate 95% confidence interval for $\acute{\boldsymbol{\mu}}$. (2)

[10]

Q. 4) A mobile phone manufacturer is considering a proposal to provide full warranty on high-end phones whereby it will undertake to replace new handsets for up to one year, against any manufacturing defects. Each phone costs Rs 25,000 to replace and the management has asked you to estimate the total cost of this warranty.

A customer services executive collected the contact information of buyers of 100 phones that were sold on 1 January and contacted them at the end of each month for one year. You have the following extract from her call journals:

- 31/01 – No defects reported, 7 customers could not be reached due to incorrect contact details
- 28/02 – 1 manufacturing defect reported, 2 customers requested not to be disturbed in future
- 31/03 – No defects reported

- 30/04 – No defects reported, 2 more customers requested not to be disturbed in future
- 31/05 – 2 manufacturing defects reported, 3 customers changed their contact details and could not be reached
- 30/06 – On leave (no calls made)
- 31/07 – No defects reported for the previous two months
- 31/08 – 2 defects reported, one of which is non manufacturing defect and hence is not covered under warranty
- 30/09 – No defects reported
- 31/10 – No defects reported
- 30/11 – Two more manufacturing defects reported
- 31/12 – No defects reported

i) Use this data to compute the Nelson-Aalen estimate of the cumulative hazard function, $\Lambda(t)$, where t is the time since purchase of a new phone and thus deduce an estimate of the survival probability beyond one year. (4)

ii) The manufacturing company sells 10,000 phones on average each year. Based on your answer in part 1, calculate the expected cost of providing the warranty over one year and construct a 95% confidence interval for the same. (4)

iii) The Finance Director believes that the estimate of the expected cost that you have calculated is too high. He notes that only 6 out of 100 phones in the investigation reported manufacturing defects.

Therefore, the expected cost of providing a warranty against manufacturing defects should be 6% * 10,000 phones * Rs25,000 = Rs1.5 crores. Explain why your estimate should be higher than this. (3)

[11]

Q. 5) **i)** State whether the following statements are True or False:

a) It is possible to have a Markov chain that has more than one stationary distribution.

b) It is possible to have a Markov chain that has no stationary distribution.

c) A Markov chain with a finite state space has at least one stationary probability distribution.

d) An irreducible Markov chain with a finite state space has a unique stationary probability distribution. (2)

ii) Define the term stationary distribution for a Markov chain with transition matrix P. (1)

iii) How many stationary distributions exist for the Markov chain with the transition matrix P given below? Find all stationary distributions for this chain:

$$P = \begin{bmatrix} 1/8 & 1/2 & 3/8 \\ 0 & 1/4 & 3/4 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

(6)

iv) Discuss in brief, how you would validate whether the stationary distribution(s) you have obtained are correct or not. (2)

[11]

Q. 6) **i)** Explain why crude mortality rates are graduated. What is the main limitation in mortality investigations which graduation cannot overcome? (4)

ii) A specialist insurer sells term assurance policies to high net-worth individuals via a dedicated bancassurance channel. The insurance company has recently carried out a mortality investigation. You are provided with the following data from the investigation:

Age	Exposed to risk	Number of deaths	Standard table mortality rates	Graduated mortality rates
x	E_x	θ_x	q_x^s	$\dot{q}_x$
40	50,000	87	0.2053 %	0.1712 %
41	48,560	84	0.2247 %	0.1920 %
42	47,190	101	0.2418 %	0.2117 %
43	44,100	112	0.2602 %	0.2332 %
44	43,600	123	0.2832 %	0.2597 %
45	40,400	110	0.3110 %	0.2917 %
46	37,280	108	0.3438 %	0.3297 %
47	35,370	122	0.3816 %	0.3738 %
48	32,100	150	0.4243 %	0.4245 %
49	29,000	139	0.4719 %	0.4820 %
50	26,200	151	0.5244 %	0.5465 %

The life company uses Indian Assured Lives Mortality (1994 -96) (modified) Ultimate table as the reference standard table.

a) Carry out the cumulative deviations test to determine whether the crude mortality rates from the investigation conform to the standard mortality rates. You must state clearly any approximations you use. (4)

b) Comment on your conclusion in part (a) above and provide two possible explanations as to why this might be the case. (2)

c) The life company carries out a graduation for the crude rates using the standard table and applies the following algorithm:

$$\dot{q}_x = 2.0844\% \cdot x \cdot q_x^s$$

The graduated rates are supplied in the table above. Comment on the appropriateness of using cumulative deviations test to determine whether the graduated rates are an acceptable reflection of the crude rates from the investigation. (3)

d) Suggest and carry out an alternative test if you were checking for overall bias in the graduated rates. Based on your results of the test or otherwise, comment on the quality of the graduation. (3)

[16]

Q. 7) **i)** Sketch the general shape of the hazard function for the following commonly used models, and give one example of a situation in which the hazard function may be expected to follow the shape you have sketched.

a) Weibull

b) Gompertz-Makenham

c) Log-logistic (6)

ii) You are given the following information from a mortality investigation of lives observed between exact ages 40 and 41:

Life	Age at Entry	Age at Exit	Cause of Exit
1	40	41	Censored
2	40	40.7	Censored
3	40.2	40.8	Censored
4	40.6	40.9	Death
5	40	40.4	Death

Assuming a constant hazard rate over (40, 41) derive an expression for the total likelihood of these observations. (4)

iii) Use your answer in part (ii) to calculate the maximum likelihood estimate of the hazard rate. Hence, obtain the MLE of probability of death at age $40, \hat{q}_{40}$. (4)

[14]

Q. 8) Each patent application (state F at filing) arriving at the Geneva office of patent registration, waits for an average five days before being classified by a Patent Officer into one of the following:

- Seeking more information from the applicant (state I)
- Straightforward application (a clear case, either way, of grant or rejection – state R)
- Complex application, warrants a further investigation by a technical expert (state E)

Generally, 40% of the applications are found to be incomplete and results in more information being sought from the applicants. 40% applications are found to be sufficiently complex so as to warrant a further investigation by a technical expert. About 80% of the cases referred to a technical expert result in further information being sought from the applicants. Only 20% of the cases result in a clear grant/reject recommendation by a technical expert. A technical expert typically takes on average 4 days to process a case referred to him/her.

In the instances where further information is sought from the applicant, whether by a Patent Officer or a Technical Expert, the applicant usually takes on average 10 days to provide additional information. The applicant is required to submit this information to an in-house panel (state P) which then makes a clear grant/reject recommendation in about 10 days on an average.

It is suggested that a time-homogenous Markov process with states F, I, E, P and R could be used to model the progress of applications through the patents approval process.

i) Draw the transition diagram and write down the generator matrix for such a Markov process. (4)

ii) Calculate the proportion of the applications where more information is sought from the applicant. (1)

iii) Derive an expression for the probability that an application is waiting with a Patent Officer for classification at time *t*, using a forward differential equation. (5)

iv) Write down the backward integral equation for the transition probability $P_{FI}(t)$ and hence show that:

$$P_{FI}(t) = 0.08 \int_0^t e^{-0.2w} P_{EI}(t-w)\ dw + 0.08\, e^{-0.1t} \int_0^t e^{-0.1w} dw$$ (3)

v) Using the expression in (iv), show that:

$$P_{FI}(t) = \frac{56}{30} e^{-0.10t} - 4e^{-0.20t} + \frac{32}{15} e^{-0.25t}$$ (6)

vi) Briefly state one way by which you can validate the closed form solution for $P_{FI}(t)$ above. (1)

[20]

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