Correct Answers are ticked in green
on the answer sheet at the end of this file.

MTH

GS-2012 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 11, 2011**
Duration : Two hours (2 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine-readable. Read the instructions given on the reverse of the answer sheet before you start filling it up. Use only HB pencils to fill-in the answer sheet.

2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. **Each corect answer will get 1 mark; each wrong answer will get a -1 mark, and a question not answered will not get you any mark.** Do not mark more than one circle for any question : this will be treated as a wrong answer.

3. There are forty (40) questions divided into four parts, Part-A, Part-B, Part-C and Part-D. Each Part consists of 10 True-False questions.

4. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

8. See the back of this page for Notation and Conventions used in this test.

NOTATION AND CONVENTIONS

$\mathbb{N}$:= Set of natural numbers

$\mathbb{Z}$:= Set of integers

$\mathbb{Q}$:= Set of rational numbers

$\mathbb{R}$:= Set of real numbers

$\mathbb{C}$:= Set of complex numbers

$\mathbb{R}^n$:= n-dimensional vector space over $\mathbb{R}$

$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$

For a differentiable real valued function $f : \mathbb{R} \to \mathbb{R}$ f' denotes its derivative and $f^{(k)}$ means the k^{th} derivative.

Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology and the metric.

INSTRUCTIONS

THERE ARE 4 PARTS AND 40 QUESTIONS IN TOTAL, CONSISTING OF 10 QUESTIONS IN EACH PART.

EVERY CORRECT ANSWER CARRIES +1 MARK AND EVERY WRONG ANSWER CARRIES −1 MARK.

PART A

1. If H_1 & H_2 are subgroups of a group G then $H_1.H_2 = \{h_1h_2 \in G | h_1 \in H_1, h_2 \in H_2\}$ is a subgroup of G.

2. There exist polynomials $f(x)$ and $g(x)$, with complex coefficients, such that $\left(\frac{f(x)}{g(x)}\right)^2 = x$.

3. Let f be real valued, differentiable on (a, b) and $f'(x) \neq 0$ for all $x \in (a, b)$. Then f is $1 - 1$.

4. The inequality $\sum_{n=0}^{\infty} \frac{(\log\log 2)^n}{n!} > \frac{3}{5}$ holds.

5. Every subgroup of order 74 in a group of order 148 is normal.

6. Let u_1, u_2, u_3, u_4 be vectors in $\mathbb{R}^2$ and
$$u = \sum_{j=1}^{4} t_j u_j \quad ; \quad t_j > 0 \text{ and } \sum_{j=1}^{4} t_j = 1.$$
Then three vectors $v_1, v_2, v_3 \in \mathbb{R}^2$ may be chosen from $\{u_1, u_2, u_3, u_4\}$ such that
$$u = \sum_{j=1}^{3} s_j v_j, \quad s_j \geq 0, \quad \sum_{j=1}^{3} s_j = 1.$$

7. The inequality
$$\sqrt{1+x} < 1 + x/2$$
for $x \in (-1, 10)$ is true

8. If n is not a multiple of 23 then the remainder when n^{11} is divided by 23 is $\pm 1 (\mod 23)$.

9. Suppose A is a nilpotent matrix and I is the identity matrix. Then $(I + A)$ is invertible.

10. The equations
$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 &= 1 \\ x_1 + \frac{1}{4}x_2 + \frac{1}{9}x_3 &= 1 \\ x_1 + \frac{1}{8}x_2 + \frac{1}{27}x_3 &= 1 \end{aligned}$$
has no solution.

PART B

11. The automorphism group Aut $(\mathbb{Z}/2 \times \mathbb{Z}/2)$ is abelian

12. Let V be the vector space of consisting of polynomials of $\mathbb{R}[t]$ of deg ≤ 2. The map $T : V \to V$ sending $f(t)$ to $f(t) + f^{'}(t)$ is invertible.

13. The polynomials $(t-1)(t-2), (t-2)(t-3), (t-3)(t-4), (t-4)(t-6) \in \mathbb{R}[t]$ are linearly independent.

14. $A \in M_2(\mathbb{C})$ and A is nilpotent then $A^2 = 0$.

15. Let P be an $n \times n$ matrix whose row sums equal 1. Then for any positive integer m the row sums of the matrix P^m equal 1.

16. There is a non trivial group homomorphism from C to R.

17. If the equation
$$xyz = 1$$
holds in a group G, does it follow that
$$yzx = 1.$$

18. Any 3×3 and 5×5 skew-symmetric matrices have always zero determinants.

19. The rank of the matrix
$$\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$$
is 2.

20. The number 2 is a prime in $\mathbb{Z}[i]$

PART C

21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then the derivative $\frac{\partial^2 f}{\partial x \partial y}$ can exist without $\frac{\partial f}{\partial x}$ existing.

22. If f is continuous on $[0, 1]$ and if $\int_0^1 f(x)x^n \, dx = 0$ for $n = 0, 1, 2, 3, \cdots$. Then $\int_0^1 f^2(x) \, dx = 0$.

23. Suppose that $f \in \mathfrak{L}^2(\mathbb{R})$. Then $f \in \mathfrak{L}^1(\mathbb{R})$.

24. The integral

$$\int_{-\infty}^{+\infty} \frac{e^{-x}}{1 + x^2} \, dx$$

is convergent.

25. If $A \subset \mathbb{R}$ and open then the interior of the closure $\overline{A}^0$ is A.

26. If $f \in C^\infty$ and $f^{(k)}(0) = 0$ for all integer $k \geq 0$, then $f \equiv 0$.

27. Let $f : [0, 1] \to [0, 1]$ be continuous then f assumes the value $\int\limits_0^1 f^2(t)dt$ somewhere in $[0, 1]$.

28. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{h}$$

exists for all $x \in \mathbb{R}$. Then f is differentiable in $\mathbb{R}$.

29. The functions $f(x) = x|x|$ and $x|\sin x|$ are not differentiable at $x = 0$.

30. The composition of two uniformly continuous functions need not always be uniformly continuous.

PART D

31. $f : [0, \infty] \to [0, \infty]$ is continuous and bounded then f has a fixed point.

32. The polynomial $X^8 + 1$ is irreducible in $\mathbb{R}[X]$.

33.

$$\text{The matrix} \quad \begin{pmatrix} 1 & \pi & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{is diagonalisable}$$

34. If a rectangle $R := \{(x, y) \in \mathbb{R}^2 \mid A \leq x \leq B, C \leq y \leq D\}$ can be covered (allowing overlaps) by 25 discs of radius 1 then it can also be covered by 101 discs of radius $\frac{1}{2}$.

35. Given any integer $n \geq 2$, we can always find an integer m such that each of the $n - 1$ consecutive integers $m + 2$, $m + 3$,..., $m + n$ are composite.

36.

$$\text{The } 10\times10 \text{ matrix} \begin{pmatrix} v_1w_1 & \cdots & v_1w_{10} \\ v_2w_1 & \cdots & v_2w_{10} \\ v_{10}w_1 & \cdots & v_{10}w_{10} \end{pmatrix} \text{ has rank 2, where } v_i, w_i \in \mathbb{C}.$$

37. If every continuos function on $X \subset \mathbb{R}^2$ is bounded, then X is compact.

38. The graph of $xy = 1$ is $\mathbb{C}^2$ is connected.

39. If $z_1, z_2, z_3, z_4 \in \mathbb{C}$ satisfy $z_1 + z_2 + z_3 + z_4 = 0$ and $|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1,$ then the least value of $|z_1 - z_2|^2 + |z_1 - z_4|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2$ is 2.

40. Consider the differential equations (with y is a function of x)

$$(1) \quad \begin{aligned} \frac{dy}{dx} &= y \\ y(0) &= 0 \end{aligned} \qquad (2) \quad \begin{aligned} \frac{dy}{dx} &= |y|^{\frac{1}{3}} \\ y(0) &= 0. \end{aligned}$$

Then (1) has infinitely many solutions but (2) has finite number of solutions.

GS-2012 (MATHEMATICS)

ANSWER SHEET

Please see reverse for instructions on filling of answer sheet.

Name	
Ref Code	
Address	
Phone	
Email	

Reference Code :

1	○	○	○	○	○
2	○	○	○	○	○
3	○	○	○	○	○
4	○	○	○	○	○
5	○	○	○	○	○
6	○	○	○	○	○
7	○	○	○	○	○
8	○	○	○	○	○
9	○	○	○	○	○
0	○	○	○	○	○

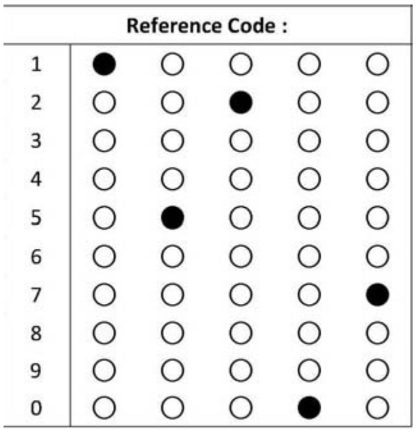
PART-A			**PART-B**			**PART-C**			**PART-D**		
	True	**False**		**True**	**False**		**True**	**False**		**True**	**False**
1	○	✓	1	○	✓	1	✓	○	1	✓	○
2	○	✓	2	✓	○	2	✓	○	2	○	✓
3	✓	○	3	○	✓	3	○	✓	3	✓	○
4	✓	○	4	✓	○	4	○	✓	4	✓	○
5	✓	○	5	✓	○	5	○	✓	5	✓	○
6	✓	○	6	✓	○	6	○	✓	6	○	✓
7	○	✓	7	✓	○	7	○	✓	7	✓	○
8	✓	○	8	✓	○	8	○	✓	8	✓	○
9	✓	○	9	✓	○	9	○	✓	9	✓	○
10	○	✓	10	○	✓	10	○	✓	10	○	✓

Signature of the Student

INSTRUCTIONS

The Answer Sheet is machine-readable. Apart from filling in the details on the answer sheet, please make sure that the Reference Code is filled by blackening the appropriate circles in the box provided on the right-top corner. <u>Only use HB pencils to fill-in the answer sheet.</u>

e.g. if your reference code is 15207 :



Also, the multiple choice questions are to be answered by blackening the appropriate circles as described below

e.g. if your answer to question 1 is (b)
and your answer to question 2 is (d)
then ....