12/1

2006

MATHEMATICS

Paper 1

Time : 3 Hours] *[Maximum Marks : 300*

INSTRUCTIONS

Candidates should attempt ***all*** *the questions in Parts A, B & C. However, they have to choose only* ***three*** *questions in Part D. The number of marks carried by each question is indicated at the end of the question.*

Answers must be written in English.

This paper has four parts :

A	20 marks
B	100 marks
C	90 marks
D	90 marks

Marks allotted to each question are indicated in each part.

SEAL

PART A

$4 \times 5 = 20$

Each question carries 5 marks.

1. (a) If $W_1, W_2, \ldots, W_n$ are subspaces of a finite dimensional vector space V such that

$$V = W_1 + W_2 + \ldots + W_n \text{ and } \dim V = \dim W_1 + \ldots + \dim W_n,$$

then prove that $V = W_1 \oplus W_2 \oplus \ldots \oplus W_n$.

(b) The points A(1, 2) and B(3, —4) are two vertices of the rectangle ABCD. The point P(3, 8) lies on CD produced. Find the coordinates of C and D.

(c) Solve $\frac{dy}{dx} - x \tan(y - x) = 1$.

(d) If f is a continuously differentiable function in a region V bounded by a closed surface Σ, then show that

$$\iiint_V \nabla f \, dV = \iint_\Sigma f \hat{n} \, d\Sigma$$

PART B

10×10=100

Each question carries 10 marks.

2. Prove that a subset B of a vector space V over a field F is a basis for V if and only if any mapping of B into any vector space W over F can be uniquely extended to a linear transformation of V into W.

3. Let W be the subspace of $\mathbf{R}^5$ spanned by the vectors

$$\alpha_1 = (2, -2, 3, 4, -1),\ \ \alpha_2 = (-1, 1, 2, 5, 2)$$

$$\alpha_3 = (0, 0, -1, -2, 3),\ \ \alpha_4 = (1, -1, 2, 3, 0).$$

Determine the space of all linear functionals f on $\mathbf{R}^5$ for which $f(\alpha_i) = 0$ for $1 \le i \le 4$. Find a basis for this space.

4. (a) Find $\lim\limits_{x \to 0} \dfrac{\tan x + 4 \tan 2x - 3 \tan 3x}{x^2 \tan x}$.

(b) Find the values of a, b, c so that

$$\lim_{x \to 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$$

5. Evaluate the following :

(a) $\displaystyle\int \frac{(\cos 2x)^{1/2}}{\sin x}\, dx$

(b) $\displaystyle\int \sqrt{\frac{1 - \sqrt{x}}{1 + \sqrt{x}}}\, dx$

6. A triangle has the lines $y = m_1 x$ and $y = m_2 x$ as two of its sides, with m_1 and m_2 being roots of the equation $bx^2 + 2hx + a = 0$. If H(a, b) is the orthocentre of the triangle, determine the equation of the third side.

7. Show that the common tangents to the circles

$$x^2 + y^2 - 6x = 0 \ \text{ and } \ x^2 + y^2 + 2x = 0$$

form an equilateral triangle.

[Turn over

8. Solve the following differential equations :

(a) $(x^2 - y^2)\,dx = 2xy\,dy$

(b) $x \cos x \frac{dy}{dx} + (x \sin x + \cos x)y = 1$

9. Prove that $\iiint\limits_D x^l y^m z^n (1 - x - y - z)^k \,dx\,dy\,dz = \frac{\lfloor l \lfloor m \lfloor k \lfloor n}{\lfloor l + m + n + k + 3}$

where l, m, n, k are positive integers and

$D = \{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0 \text{ and } x + y + z \leq 1\}$.

10. With the usual notation, prove that

$$\left\{ \begin{matrix} i \\ i\ j \end{matrix} \right\} = \frac{\partial}{\partial x^j} (\log \sqrt{g})$$

11. (a) A solid displaces $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ of its volume respectively when it floats in three different liquids. Find the volume it displaces when it floats in a mixture formed of equal volumes of liquids.

(b) A quadrant of a circle is just immersed vertically in a heavy homogeneous liquid, with one edge in the surface. Find the centre of pressure.

PART C

6×15=90

Each question carries 15 marks.

12. Let V and W be vector spaces over a field F and dim V < ∞. For any linear transformation T from V into W, prove that

$$\text{rank}\,(T) + \text{nullity}\,(T) = \dim V$$

and deduce that, for any m × n matrix A over F,

$$\text{row rank}\,(A) = \text{column rank}\,(A).$$

13. Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} -1 & \text{if } x = 1 \\ (x-1)^2 \sin \dfrac{1}{x-1} - |x|, & \text{if } x \neq 1. \end{cases}$$

Determine the set of points at which f is not differentiable.

14. A cone is circumscribed about a sphere of radius r. Find its semi-vertical angle for the volume of the cone to be minimum.

15. (a) One end of a uniform rod is attached to a hinge and the other end is supported by a string attached to the extremity of the rod, and the rod and the string are inclined at the same angle θ to the horizontal. If w is the weight of the rod, show that the action at the hinge is

$$\frac{w}{4}\sqrt{8 + \operatorname{cosec}^2 \theta}\,.$$

(b) A body of 65 kg force is suspended by two strings of lengths 5 and 12 metres attached to two points in the same horizontal line whose distance apart is 13 metres. Find the tensions of the strings.

[Turn over

16. (a) In a right-angled triangle, the lengths of the sides are a and b $(0 < a < b)$. Prove that the radius of the circle passing through the midpoint of the smaller side and touching the hypotenuse at its midpoint is

$$b\sqrt{a^2 + b^2} \bigg/ 4a .$$

(b) Find an equation of the circle passing through (1, 0) and (0, 1) and having the smallest possible radius.

17. Solve the equations :

(a) $y' = \dfrac{3x^2 - 2xy}{x^2 - 2y}$

(b) $(xy^3 + y)\,dx = -2\,(x^2y^2 + xy^4)\,dy$

PART D

$3 \times 30 = 90$

*Answer any **three** of the following questions. Each question carries 30 marks.*

18. (a) Define the concept of the minimal polynomial of an $n \times n$ matrix over a field F and prove that any two similar polynomials have the same minimal polynomial.

(b) Find the minimal polynomial, the characteristic matrices and the characteristic values of the matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

(c) State and prove the Cayley – Hamilton theorem.

19. (a) Evaluate $\int x^2 \log(1 - x^2)\, dx$ and deduce that

$$\frac{1}{1 \times 5} + \frac{1}{2 \times 7} + \frac{1}{3 \times 9} + \ldots = \frac{8}{9} - \frac{2}{3} \log 2$$

(b) The line $y = mx$ bisects the area enclosed by the lines $x = 0$, $y = 0$, $x = 3/2$ and the curve $y = 4x + 1 - x^2$. Find the value of m.

(c) Evaluate $\int_0^1 \frac{dx}{(1 + x)(2 + x)\sqrt{x(1 - x)}}$.

20. (a) Prove that in general, four normals can be drawn to an ellipse from any point.

(b) Prove that the eccentricity of the ellipse in which the normal at one end of a latus rectum passes through an end of the minor axis is given by

$$e^4 + e^2 - 1 = 0.$$

(c) Any tangent to an ellipse is cut by the tangent at the extremities of the major axis at T and T′. Prove that the circle on TT′ as diameter passes through the foci.

21. (a) Solve the equation and find the general solution :

$$y'' + 4y' + 5y = x^3 + \cos 2x$$

(b) Solve $y'' + 9y = \sin 3x$ by the method of variation of parameters.

22. (a) A particle moves with a central acceleration which varies inversely as the cube of the distance. If it is projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, then show that the equation to its path is $r \cos\left(\frac{\theta}{\sqrt{2}}\right) = a$.

(b) A spherical rain-drop falling freely receives in each instant an increase of volume equal to λ times its surface at that instant. Find the velocity at the end of time t and the distance fallen through in that time.

(c) Find the amplitude and the time period of a particle moving with simple harmonic motion which has a velocity of 9 m/s and 4 m/s at the distance of 2 m and 3 m respectively from the centre.

SEAL

12/2

2006

MATHEMATICS

Paper 2

Time : 3 Hours] *[Maximum Marks : 300*

INSTRUCTIONS

*Candidates should attempt **all** the questions in Parts A, B & C. However, they have to choose only **three** questions in Part D. The number of marks carried by each question is indicated at the end of the question.*

Answers must be written in English.

This paper has four parts :

A	20 marks
B	100 marks
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Marks allotted to each question are indicated in each part.

SEAL

PART A *4×5=20*

Each question carries 5 marks.

1. (a) How many distinct (upto isomorphism) abelian groups of order 108 are there ? List all of them.

(b) Define $f : [0, 8] \to \mathbf{R}$ by $f(x) = [x]$, the largest integer less than or equal to x. Then, is f Riemann – Stieltjes integrable with respect to α, where α is defined on [0, 8] by $\alpha(x) = 2x$? If yes, evaluate $\int_0^8 f \, d\alpha$.

(c) Use Newton's divided difference interpolation formula to obtain f(9), given the tabular values :

x	3	7	11	13	17
f(x)	150	392	1452	2366	5202

(d) The following data relates to the profit earned by a few companies. Determine the Pearsons coefficient of skewness for this data and interpret your result.

Profit (in Rs. thousands)	Number of companies
10 – 12	7
12 – 14	15
14 – 16	18
16 – 18	20
18 – 20	25
20 – 22	10
22 – 24	5

PART B

10×10=100

Each question carries 10 marks.

2. Prove that the ring of Gaussian integers is a Euclidean domain.

3. Prove that any group of order 48 must have a normal subgroup of order 8 or 16.

4. Consider the functions f, g : $\mathbf{R} \to \mathbf{R}$ defined by $f(0) = 0 = g(0)$ and, for any $0 \neq x \in \mathbf{R}$,

$$f(x) = x \sin \frac{1}{x} \text{ and } g(x) = x^2 \sin \frac{1}{x}.$$

Prove that f is not differentiable at 0 and g is differentiable at all points x, but g' is not continuous.

5. Define the notion of an analytic function and state and prove the maximum modulus theorem for analytic functions.

6. (a) Form a partial differential equation by eliminating the arbitrary function from

$$F(x + y + z,\ x^2 + y^2 + z^2) = 0$$

(b) Find the integral surface of the linear partial differential equation

$$x (y^2 + z) p - y (x^2 + z) q = (x^2 - y^2) z$$

which contains the straight line $x + y = 0$; $z = 1$.

7. (a) Find the moment of inertia of a homogeneous elliptic plate $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ with respect to y-axis.

(b) A particle of mass M is projected with initial velocity u at an angle α with the horizontal. Obtain the equations of the path of the particle using Lagrange's equations of motion (neglect the air resistance).

8. Fit the following data by a cubic spline curve and find f(0·5) and f(2·5) using the end-conditions $M_1 = M_5 = 0$.

x	0	1	2	3	4
f(x)	− 8	− 7	0	19	56

9. (a) Determine the constants a and b by the method of least squares such that $y = ae^{bx}$ fits with the following data :

x	2	4	6	8	10
y	4·077	11·084	30·128	81·897	222·620

(b) The ranks obtained by a set of 10 students in Mathematics test (variable X) and Physics test (variable Y) are given below. Determine the Spearman's rank correlation of this data :

Rank for X	1	2	3	4	5	6	7	8	9	10
Rank for Y	3	1	4	2	6	9	8	10	5	7

10. (a) A and B throw a pair of dice alternately. A wins if he throws 6 before B throws 7. B wins if he throws 7 before A throws 6. If A begins the game, find his chances of winning.

(b) State and prove the weak law of large numbers.

11. Using Charnes penalty (Big M) method, solve the LPP :

Minimize $z = 4x_1 + 3x_2$

subject to the following

$$2x_1 + x_2 \geq 10$$

$$-3x_1 + 2x_2 \leq 6$$

$$x_1 + x_2 \geq 6$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0$$

PART C *6×15=90*

Each question carries 15 marks.

12. If f is a continuous complex function on [a, b], prove that there exists a sequence $\{P_n\}$ of polynomials such that $P_n(x) \to f(x)$ uniformly on [a, b].

13. Let f be a complex valued function defined on a region and suppose f has an isolated singularity at a. Then prove that the point z = a is a removable singularity if and only if $\lim_{z \to a} (z - a) f(z) = 0$.

14. (a) Find a complete integral of $p^2x + q^2y = z$.

(b) Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$.

(c) Obtain D'Alembert's solution of $\frac{\partial^2 z}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 z}{\partial t^2}$.

15. (a) Test whether the motion specified by

$$\vec{V} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

is a possible motion for an incompressible fluid. If so, determine the equations of stream lines and the circulation round the unit circle with centre at origin.

(b) Air obeying Boyle's law is in motion in a uniform tube of small section. If ρ is the density and v is the velocity at a distance x from a fixed point at the time t, then prove that

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2} ((v^2 + k)\rho).$$

16. (a) Given the following data, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1·1$.

x	1·0	1·1	1·2	1·3	1·4	1·5	1·6
y	7·989	8·403	8·781	9·129	9·451	9·750	10·031

(b) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Simpson's $\frac{3}{8}$ rule and compare your result with the actual value.

17. Obtain an optimal solution to the following transportation problem in which the cell entries represent unit costs, by finding the initial basic feasible solution using Vogel's method.

	W_1	W_2	W_3	Available
F_1	2	7	4	5
F_2	3	3	1	8
F_3	5	4	7	7
F_4	1	6	2	14
Requirement	7	9	18	

PART D

3×30=90

Answer any ***three*** *of the following questions. Each question carries 30 marks.*

18. Let S_n be the group of permutations of an n-element set and A_n the set of even permutations in S_n. Prove the following :

(a) A_n is a normal subgroup of index 2 in S_n, for $n > 1$.

(b) A_n is generated by the set of all 3-cycles in S_n, for $n > 2$.

(c) The derived group of S_n is A_n.

(d) A_n is simple if and only if $n \neq 4$.

19. (a) Prove that every continuous function $f : [0, 1] \to \mathbf{R}$ is uniformly continuous.

(b) Let X be a non compact set in **R**. Prove the following :

(i) There exists a continuous function on X which is not bounded.

(ii) There exists a continuous and bounded function on X which has no maximum.

(iii) If X is bounded, there exists a continuous function on X which is not uniformly continuous.

20. State and prove Morera's theorem and deduce that any differentiable complex valued function defined on an open subset of $\mathbb{C}$ is analytic.

21. (a) Given that $\frac{dy}{dx} = 1 + y^2$ and $y(0) = 0$, find $y(0{\cdot}4)$ by taking $h = 0{\cdot}2$ and by using Runge – Kutta fourth order formula.

(b) Determine the value of y when x = 0·1, given that $y' = x^2 + y$, $y(0) = 1$ using the modified Euler formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right]$$

n = 0, 1, 2, ... by choosing $y_1^{(0)}$ from Euler's formula

$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$.

(c) Apply the Milne's predictor-corrector method to find a solution to the equation $y' = x - y^2$, $y(0) = 0$ in the range $0 \le x \le 1$.

22. (a) An airlines organisation has one reservation clerk in a branch who handles information regarding passenger reservation and flight timings. It is found that the number of customers arriving at the branch during any period is Poisson distributed with an arrival rate of eight per hour and the clerk takes six minutes on an average to serve a customer, with an exponentially distributed service time.

(i) Find the probability that the system is busy.

(ii) Find the average time that a customer spends in the system.

(iii) Find the average length of queue and the number of customers in the system.

(b) Determine the optimal sequence of jobs that minimizes the total elapsed time, based on the following information. Processing time on machines is given in hours and passing is not allowed.

Job	A	B	C	D	E	F	G
Machine M_1	3	8	7	4	9	8	7
Machine M_2	4	3	2	5	1	4	3
Machine M_3	6	7	5	11	5	6	12