1·F·S-2009

Sl. No. 8206

B-JGT-J-NBA

MATHEMATICS

Paper I

Time Allowed : Three Hours	Maximum Marks : 200

INSTRUCTIONS

Candidates should attempt questions 1 and 5 which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.

All questions carry equal marks.

Answers must be written in ENGLISH only.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless indicated otherwise, symbols & notations carry their usual meaning.

(Contd.)

Section – A

1. Answer any ***four*** of the following questions :

(a) Let V be the vector space of polynomials over R. Let U and W be the subspaces generated by

$$\left\{t^3+4t^2-t+3,\ t^3+5t^2+5,\ 3t^3+10t^2-5t+5\right\}$$

and

$$\left\{t^3+4t^2+6,\ t^3+2t^2-t+5,\ 2t^3+2t^2-3t+9\right\}$$

respectively. Find

(i) dim $(U+W)$

(ii) dim $(U\cap W)$. 10

(b) Find a linear map $T: R^3 \to R^4$ whose image is generated by (1, 2, 0, –4) and (2, 0, –1, –3). 10

(c) (i) Find the difference between the maximum and the minimum of the function $\left(a-\frac{1}{a}-x\right)\left(4-3x^2\right)$ where a is a constant and greater than zero.

(ii) If $f(h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(\theta h)$, $0<\theta<1$

Find θ, when $h=1$ and $f(x)=(1-x)^{5/2}$. 5+5=10

(d) Evaluate :

(i) $\int_0^{\pi/2} \frac{\sin^2 x\, dx}{\sin x + \cos x}$ (ii) $\int_1^{\infty} \frac{x^2 dx}{\left(1+x^2\right)^2}$. 6+4=10

(e) Show that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z = 2$ in a circle of radius unity and find the equation of the sphere which has this circle as one of its great circles. 10

2. (a) Let T be the linear operator on R^3 defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$.

(i) Show that T is invertible.

(ii) Find a formula for T^{-1}. 10

(b) Find the rank of the matrix :

$$A = \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$$ 10

(c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. Is A similar to a diagonal matrix ? If so, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. 10

(d) Find an orthogonal transformation of co-ordinates to reduce the quadratic form $q(x, y) = 2x^2 + 2xy + 2y^2$ to a canonical form. 10

3. (a) The adiabatic law for the expansion of air is $PV^{1 \cdot 4} = K$, where K is a constant. If at a given time the volume is observed to be 50 c.c. and the pressure is 30 kg per square centimetre, at what rate is the pressure changing if the volume is decreasing at the rate of 2 c.c. per second ? 10

(b) Determine the asymptotes of the curve
$x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0.$ 10

(c) Evaluate :

$$\iint_D x \sin(x+y)\, dx\, dy,$$

where D is the region bounded by

$0 \leq x \leq \pi$ and $0 \leq y \leq \frac{\pi}{2}$. 10

(d) Evaluate $\iiint (x+y+z+1)^4\, dx\, dy\, dz$ over the region defined by $x \geq 0,\ y \geq 0,\ z \geq 0$ and $x + y + z \leq 1$. 10

4. (a) Obtain the equations of the planes which pass through the point (3, 0, 3), touch the sphere $x^2 + y^2 + z^2 = 9$ and are parallel to the line $x = 2y = -z$. 10

(b) The section of a cone whose vertex is P and guiding curve is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$ by the plane $x = 0$ is a rectangular hyperbola. Show that the locus of P is

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1.$$ 10

(c) Prove that the locus of the poles of the tangent planes of the conicoid $ax^2+by^2+cz^2=1$ with respect to the conicoid $\alpha x^2+\beta y^2+\gamma z^2=1$ is the conicoid $\dfrac{\alpha^2 x^2}{a}+\dfrac{\beta^2 y^2}{b}+\dfrac{\gamma^2 z^2}{c}=1$. 10

(d) Show that the lines drawn from the origin parallel to the normals to the central conicoid $ax^2+by^2+cz^2=1$ at its points of intersection with the planes $lx+my+nz=p$ generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2.$$ 10

Section – B

5. Answer any *four* :

(a) Solve :

$$\sec^2 y\,\frac{dy}{dx}+2x\tan y=x^3.$$ 10

(b) Find the 2nd order ODE for which e^x and x^2e^x are solutions. 10

(c) A uniform rectangular board, whose sides are $2a$ and $2b$, rests in limiting equilibrium in contact with two rough pegs in the same horizontal line at a distance d apart. Show that the inclination θ of the side $2a$ to the horizontal is given by the equation

$$d\cos\lambda\left[\cos(\lambda+2\theta)\right]=a\cos\theta-b\sin\theta$$

where λ is the angle of friction. 10

(d) A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being μ and μ'. The particle is slightly displaced towards one of them, show that the time of small oscillation is

$$\frac{2\pi}{\sqrt{(\mu+\mu')}}.$$ 10

(e) Verify Green's theorem in the plane for $\oint_C \left[\left(xy+y^2\right)dx + x^2 dy\right]$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$. 10

6. (a) Solve :

$$\left(y^3-2yx^2\right)dx+\left(2xy^2-x^3\right)dy=0.$$ 10

(b) Solve :

$$\left(\frac{dy}{dx}\right)^2-2\frac{dy}{dx}\cos hx+1=0.$$ 8

(c) Solve :

$$\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y=x^2e^{-x}.$$ 10

(d) Show that e^{x^2} is a solution of

$$\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+\left(4x^2-2\right)y=0.$$

Find a second independent solution. 12

7. (a) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the sphere is in contact. If θ and ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that

$$\tan \phi = \frac{3}{8} + \tan \theta. \qquad 10$$

(b) A particle moves with a central acceleration $\mu\left(\gamma + \frac{a^4}{\gamma^3}\right)$ being projected from an apse at a distance a with a velocity $2\sqrt{\mu}\, a$.

Prove that its path is

$$\gamma^2\left(2 + \cos\sqrt{3}\,\theta\right) = 3a^2. \qquad 10$$

(c) A shell, lying in a straight smooth horizontal tube, suddenly explodes and breaks into portions of masses m and m'. If d is the distance apart of the masses after a time t, show that the work done by the explosion is

$$\frac{1}{2}\,\frac{mm'}{m+m'}\cdot\frac{d^2}{t^2}. \qquad 10$$

(d) A hollow conical vessel floats in water with its vertex downwards and a certain depth of its axis immersed. When water is poured into it up to the level originally immersed, it sinks till its mouth is on a level with the surface of the water. What portion of axis was originally immersed ? 10

8. (a) Show that

$$\bar{A} = \left(6xy + z^3\right)\hat{i} + \left(3x^2 - z\right)\hat{j} + \left(3xz^2 - y\right)\hat{k}$$

is irrotational. Find a scalar function ϕ such that $\bar{A} = \text{grad}\,\phi$. 10

(b) Let $\psi(x, y, z)$ be a scalar function. Find grad ψ and $\nabla^2\psi$ in spherical coordinates. 8

(c) Verify Stokes' theorem for

$$\bar{A} = (y - z + 2)\hat{i} + (yz + 4)\hat{j} - xz\hat{k},$$

where S is the surface of the cube $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 2$, $z = 2$ above the xy-plane. 12

(d) Show that, if $\bar{r} = x(s)\,\hat{i} + y(s)\,\hat{j} + z(s)\,\hat{k}$ is a space curve, $\frac{d\bar{r}}{ds} \cdot \frac{d^2\bar{r}}{ds^2} \times \frac{d^3\bar{r}}{ds^3} = \frac{\tau}{\rho^2}$, where τ is the torsion and ρ the radius of curvature. 10