Entrance Examination, 2004

M.Sc. (Mathematics/Applied Mathematics)

Hall Ticket No.							

Time: 2 hours

Max. Marks: **100**
Part A: **25**
Part B: **75**

Instructions

1. Calculators are not allowed.
2. Part A carries 25 marks. Each correct answer carries **1 mark** and each wrong answer carries $-\frac{1}{4}$ mark. So do not gamble. If you want to change any answer, cross out the old one and circle the new one. Over written answers will be ignored.
3. Part B carries 75 marks. Instructions for answering Part B are given at the beginning of Part B.
4. Do not detach any pages from this answer book. It contains 16 pages. Pages 15 and 16 are for rough work.

Answer Part A by **circling** the correct letter in the array below:

1	a	b	c	d	e
2	a	b	c	d	e
3	a	b	c	d	e
4	a	b	c	d	e
5	a	b	c	d	e
6	a	b	c	d	e
7	a	b	c	d	e
8	a	b	c	d	e
9	a	b	c	d	e
10	a	b	c	d	e
11	a	b	c	d	e
12	a	b	c	d	e
13	a	b	c	d	e
14	a	b	c	d	e
15	a	b	c	d	e
16	a	b	c	d	e
17	a	b	c	d	e
18	a	b	c	d	e
19	a	b	c	d	e
20	a	b	c	d	e
21	a	b	c	d	e
22	a	b	c	d	e
23	a	b	c	d	e
24	a	b	c	d	e
25	a	b	c	d	e

PART A

Find the correct answer and mark it on the answer sheet on the **top page**. A correct answer gets **1 mark** and a wrong answer gets a $-(1/4)$ **mark**.

1. In a cyclic group of order 36, the number of subgroups of order 6 is
   (a) 1. (b) 2. (c) 6. (d) 8. (e) none of these.

2. (I) The zero vector cannot be a vector in a basis of $\mathbb{R}^5$.
   (II) There are 4 linearly independent vectors in $\mathbb{R}^5$.
   (III) There are 7 vectors in $\mathbb{R}^5$ which generate $\mathbb{R}^5$.

   (a) All the above three statements are false.
   (b) All the above three statements are true.
   (c) I and II are true but III is false.
   (d) I and III are true but II is false.
   (e) II and III are true but I is false.

3. In the group Z_{12} of integers modulo 12 under addition, the order of $\bar{5}$ is
   (a) 5. (b) 7. (c) 8. (d) 12. (e) none of these.

4. Let $f : [0,1] \to [0,1]$ be a continuous function. Then

   (a) f must be increasing.
   (b) f must have a fixed point
   (c) f must be decreasing.
   (d) f must be onto.
   (e) none of the above.

5. A is a 3×3 real matrix whose characteristic polynomial is $X^3 - X^2 - 3X + 3$. then

   (a) $A^3 - A^2 - 3A + 3I = I$.
   (b) The eigen values of A are integers.
   (c) A is not diagonalizable.
   (d) A has only one eigen value.
   (e) none of the above.

6. If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $(S^{-1}AS)^{50} =$

(a) $\begin{pmatrix} 1 & 50 \\ 0 & 1 \end{pmatrix}$. (b) $\begin{pmatrix} 1 & -50 \\ 0 & 1 \end{pmatrix}$. (c) $\begin{pmatrix} 1 & 0 \\ 50 & 1 \end{pmatrix}$. (d) $\begin{pmatrix} 1 & 0 \\ -50 & 1 \end{pmatrix}$.
(e) none of the above.

7. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2x, & \text{if } x \text{ is rational} \\ 3 - x, & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is continuous
(a) only at $x = 1$.
(b) only at $x = 2$.
(c) only at $x = 0$.
(d) at all points of $\mathbb{R}$.
(e) at no point of $\mathbb{R}$.

8. Define the function $f : [-1, 1] \to \mathbb{R}$ by $f(x) = x^2 - [x]$, where $[x]$ is the greatest integer less than or equal to x. Then

(a) $\lim_{x\to 0^-} f(x)$ exists but not $\lim_{x\to 0^+} f(x)$.
(b) $\lim_{x\to 0^+} f(x)$ exists but not $\lim_{x\to 0^-} f(x)$.
(c) both $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ exist and are equal.
(d) both $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ exist, and are not equal.
(e) none of the above limits exist.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x + 1| - |x|$, then the range of f is
(a) the whole of $\mathbb{R}$.
(b) $\{1\}$.
(c) the closed interval $[-1, 1]$.
(d) the unbounded subset of $\mathbb{R}$.
(e) none of the above.

10. Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(1) = 1, f(2) = 5$ and $f(3) = -4$. Which one of these follows from the intermediate value theorem?

(a) $f(x) = 0$ for some x between 1 and 2.
(b) $f(x) + 1 = 0$ for some x between 2 and 3.
(c) $f(x) + 5 = 0$ for some x between 1 and 3.
(d) $f(x) = 0$ for some $x > 3$.
(e) none of the above.

11. $f(x) = x^2|x|$ on $x \in (-1, 1)$.

(a) f is differentiable everywhere except at $x = 0$.
(b) f is not differentiable at $x = 0$ but continuous.
(c) f' is not continuous at $x = 0$.
(d) f'' is not continuous at $x = 0$.
(e) none of the above.

12. The area enclosed by the ellipse $5x^2 + 5y^2 - 6xy + 22x - 26y + 35 = 0$ is
(a) $\frac{\pi}{5}$. (b) $\frac{\pi}{4}$. (c) $\frac{\pi}{2}$. (d) π. (e) none of these.

13. How may subsets A of $\{1, 2, 3, 4\}$ are there that $(A \cup \{1, 2\}) - (A \cap \{1, 2\})$ has exactly one point?
(a) 1. (b) 2. (c) 3. (d) 4. (e) zero.

14. How many zeroes are there for the function $x^2 - 5|x| + 6$?
(a) 2. (b) 3. (c) 4. (d) 6. (e) none of these.

15. Which of these subsets of the plane is a bounded set?

(a) $\{(x, y) \in I\!R^2 \ / \ x^2 + y^2 < 10\}$.
(b) $\{(x, y) \in I\!R^2 \ / \ 1 < \frac{x^2}{4} + \frac{y^2}{9}\}$.
(c) $\{(x, y) \in I\!R^2 \ / \ x = y\}$.
(d) $\{(x, y) \in I\!R^2 \ / \ y^2 < 4x\}$.
(e) none of the above.

16. If the circles $x^2 + y^2 = 1$ and $(x - a)^2 + (y - b)^2 = 1$ have exactly one point in common, then (a, b) lies on

(a) the first circle $x^2 + y^2 = 1$.
(b) the second circle $(x - a)^2 + (y - b)^2 = 1$.
(c) the circle $x^2 + y^2 = 2$.
(d) the circle $x^2 + y^2 = 4$.
(e) none of the above.

17. If ω is the cube root of unity, then the value of the determinant

$$\begin{vmatrix} 2 & 2\omega & 2\omega^2 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

is
(a) 1. (b) −1. (c) 0. (d) −2. (e) 2.

18. The equation of the plane through the point $(-1, 3, 2)$ and perpendicular to each of the planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$ is

(a) $2x - 4y + 3z + 8 = 0$.
(b) $2x - 4y - 3z + 8 = 0$.
(c) $2x - 4y - 3z - 8 = 0$.
(d) $2x - 4y + 3z - 8 = 0$.
(e) $2x + 4y + 3z - 16 = 0$.

19. If three consecutive coefficients of the binomial expansion of $(1+x)^n$ are 36, 84 and 126 respectively, then the value of n is

(a) 18. (b) 5. (c) 12. (d) 9. (e) 6.

20. To obtain a basis of $I\!R^4$ one needs to take along with the vectors $(1, 1, -1, -1)$, $(1, 0, 1, 0)$, $(0, 1, 0, 1)$ the vector

(a) $(1, 1, 0, 0)$. (b) $(0, 0, 1, 1)$.
(c) $(0, 1, 1, 2)$. (d) $(1, 1, 0, 2)$.
(e) $(1, 1, 0, 1)$.

21. The series $\left\{\sum \frac{x^n}{n}\right\}$

(a) converges for all $x \in I\!R$.
(b) converges for $x \in [-1, 0]$.
(c) converges for $x \in [0, 1]$.
(d) converges for $x \in [-1, 1]$.
(e) none of the above.

22. If $f(x)g(x)$ is continuous on (a, b), then

    (a) both f and g must be continuous.

    (b) both f and g must be bounded.

    (c) at least one of f and g must be bounded.

    (d) one or both of f and g can be discontinuous.

    (e) none of the above.

23. In how many ways can we arrange the letters A, B, C, D, E, F, so that A and F are adjacent?
    (a) 120. (b)144. (c) 720. (d) 360. (e) 240.

24. There are 10 balls in a bag numbered $1, 2, \ldots, 10$. A ball is drawn, replaced and again a ball is drawn. The probability that the same ball is drawn is
    (a) $\frac{1}{10}$. (b) $\frac{1}{3}$. (c) $\frac{1}{5}$. (d) $\frac{1}{2}$. (e) $\frac{1}{20}$.

25. Two numbers are picked up from $\{21, \ldots, 40\}$ without replacement. The probability that neither of them is prime is

    (a) more than 0.95.

    (b) in the interval $(0.4, 0.6]$.

    (c) in the interval $(0.6, 0.75]$.

    (d) in the interval $(0.75, 0.85]$.

    (e) in the interval $(0.85, 0.85)$.

Part - B (pages 7 to 14)

There are 15 questions in this part. Each question carries 5 marks. Answer as many as you can. The maximum you can score is 75 marks. Brief proofs are needed for each question in the place provided below the question.

1. Give an example of a relation that is symmetric and transitive but not reflexive.

2. Show that $7^{10} - 1$ is always divisible by 528.

3. Show that in an abelian group of odd order, the product of all the elements of the group is the identity element.

4. Let $\alpha_1 = (1, 0, -1, 2)$ and $\alpha_2 = (2, 3, 1, 1)$ and let W be the linear span of α_1 and α_2. Which linear functions f such that $f(x_1, x_2, x_3, x_4) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ annihilate W, i.e., for what values of the c_i's does f annihilate W?

5. Find the sum $1 + \cos x + \cos 2x + \ldots + \cos nx$.

6. If the real numbers $a_0, a_1, a_2, \ldots, a_n$, $n \geq 1$ satisfy

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \cdots + \frac{1}{n+1}a_n = 0,$$

then show that the polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ has a root in $[0, 1]$..

7. If a sequence $\{a_n\}$ of real numbers satisfies

$$|a_{n+2} - a_{n+1}| \leq c\ |a_{n+1} - a_n|$$

for all integers $n \geq 1$, where $0 < c < 1$ is a constant, then show that $\{a_n\}$ is convergent.

8. The random variable X has Binomial distribution. Its variance is half its mean and $P(X = 10) = P(X = 12)$. Compute $P(X = 4)$, $E(X)$ and $V(X)$.

9. There are 4 questions in an exam, each with two alternatives, one of which is correct. A candidate does not know the correct answers and decides to mark the alternatives randomly. Evaluate the probabilities of the following events.

   (a) (i) At most two are wrong.

   (ii) At least one is wrong.

   (b) At least one is correct, given that one is wrong.

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies the following condition

$$|f(x) - f(y)| < c|x - y|.$$

Then show that f is uniformly continuous.

11. Find the values of $p > 0$ for which the series $\sum_{n=1}^{\infty} \frac{1}{(n^p + n)}$ converges.

12. Find the derivative of the function f at $x = 0$, where

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

13. Find the eigen values and eigen vectors of

$$\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

14. Find the general solution of the differential equation

$$x^2y'' - x(x+2)y' + (x+2)y = 0.$$

15. Find a linear transformation which maps $(1,0,1)$ to $(1,1)$, $(1,1,1)$ to $(0,1)$ and $(1,0,0)$ to $(1,0)$. Find its kernel.