3.

C-47

University of Hyderabad,
Entrance Examination, 2004
Ph.D. (Mathematics/Applied Mathematics)

Hall Ticket No.							

Time: **2** hours

Max. Marks: **75**
Part A: **25**
Part B: **50**

Instructions

1. Calculators are not allowed.
2. Part A carries 25 marks. Each correct answer carries **1 mark** and each wrong answer carries $-\frac{1}{4}$ **mark.** So do not gamble. If you want to change any answer, cross out the old one and circle the new one. Over written answers will be ignored.
3. Part B carries 50 marks. Instructions for answering Part B are given at the beginning of Part B.
4. Use a separate booklet for Part B.

Answer Part A by **circling** the correct letter in the array below:

1	a	b	c	d	e
2	a	b	c	d	e
3	a	b	c	d	e
4	a	b	c	d	e
5	a	b	c	d	e
6	a	b	c	d	e
7	a	b	c	d	e
8	a	b	c	d	e
9	a	b	c	d	e
10	a	b	c	d	e
11	a	b	c	d	e
12	a	b	c	d	e
13	a	b	c	d	e
14	a	b	c	d	e
15	a	b	c	d	e
16	a	b	c	d	e
17	a	b	c	d	e
18	a	b	c	d	e
19	a	b	c	d	e
20	a	b	c	d	e
21	a	b	c	d	e
22	a	b	c	d	e
23	a	b	c	d	e
24	a	b	c	d	e
25	a	b	c	d	e

C-47

PART A (25 marks)

This part contains 25 questions. Each correct answer carries **1 mark** while a wrong answer carries **minus quarter mark.**

(1) The set $\{x \in \mathbb{R} : |x+1| = |x| + 1\}$ is same as
(a) $\{x \in \mathbb{R} : x > 0\}$.
(b) $\{x \in \mathbb{R} : x \geq 0\}$.
(c) whole $\mathbb{R}$.
(d) the empty set.
(e) none of the above.

(2) The range of the function $f(x) = \dfrac{x}{\sqrt{x^2+1}}$, $x \in \mathbb{R}$ is
(a) [-1,1]. (b) [-1,1). (c) (-1,1]. (d) (-1,1). (e) none of these.

(3) The set $\{z \in \mathbb{C} : z^3 \text{ is not a real number}\}$ is
(a) a bounded open connected subset of $\mathbb{C}$.
(b) a bounded open disconnected subset of $\mathbb{C}$.
(c) an unbounded open disconnected subset of $\mathbb{C}$.
(d) an unbounded connected but not open subset of $\mathbb{C}$.
(e) none of the above.

(4) Consider the sequence $\{a_n\}$ of real numbers where $a_1 > 1$ and $a_{n+1} = 2 - \dfrac{1}{a_n}, n \geq 1$. Then the sequence $\{a_n\}$ is
(a) bounded but not monotone.
(b) not bounded but monotone
(c) both bounded and monotone.
(d) neither bounded nor monotone.
(e) none of the above.

(5) Let f be a real valued function defined on $[a, b]$ such that f is differentiable on (a, b) and both $f(a+)$, $f(b-)$ exist. Then
(a) there is a $c \in (a, b)$ such that $(f(b-) - f(a+)) = f'(c)(b-a)$.
(b) there is a $c \in (a, b)$ such that $(f(b-) - f(a)) = f'(c)(b-a)$.
(c) there is a $c \in (a, b)$ such that $(f(b) - f(a+)) = f'(c)(b-a)$.
(d) there is a $c \in (a, b)$ such that $(f(b) - f(a)) = f'(c)(b-a)$.
(e) there is no $c \in (a, b)$ satisfying any of the above statements.

C. 47

(6) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & \text{if } x \neq 0 \text{ and } y \in \mathbb{R} \\ 0, & \text{if } x = 0 \text{ and } y \in \mathbb{R}. \text{ Then} \end{cases}$$

(a) f is continuous but not differentiable at $(0, 0)$.
(b) f is differentiable at $(0, 0)$.
(c) f has all first order partial derivatives at $(0, 0)$.
(d) f does not have all first order partial derivatives at $(0, 0)$.
(e) none of the above.

(7) The function $f(z) = |z|^4, z \in \mathbb{C}$ is
(a) not continuous at 0.
(b) continuous but not differentiable at 0.
(c) differentiable but not analytic at 0.
(d) analytic at 0.
(e) none of the above.

(8) A harmonic conjugate $v(x, y)$ of the function $u(x, y) = x^3 - 3xy^2$ on $\mathbb{C}$ is
(a) $y^3 - 3x^3$.
(b) $3x^2y - y^3$.
(c) $x^3 - 3xy^2$.
(d) $x^3 - 3x^2y$.
(e) none of the above.

(9) The residue of $f(z) = \dfrac{e^{z^3}}{(z-1)^3}$ at $z = 1$ is
(a) $\frac{9e}{2}$. (b) $\frac{15e}{2}$. (c) $\frac{21e}{2}$. (d) $\frac{27e}{2}$. (e) none of these.

(10) The value of $3^{37}(\text{mod} 481)$ is
(a) 3. (b) 36. (c) 37. (d) 13. (e) none of these.

(11) Let C_{12} be a cyclic group generated by a. The order of the element $(a^7, a^6) \in C_{12} \times C_{12}$ is
(a) 2. (b) 5. (c) 6. (d) 12. (e) none of these.

(12) The number of Sylow 7 - subgroups in a group of order 392 is
(a) 2 or 3.
(b) 3 or 7.
(c) 1 or 8.
(d) 2 or 7.
(e) none of the above.

C-47

(13) Consider the quotient ring $\mathbb{Z}/95\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. The number of elements $x \in \mathbb{Z}/95\mathbb{Z}$ such that $x^2 = \bar{1}$ is

(a) 1. (b) 2. (c) 3. (d) 4. (e) none of these.

(14) The number of proper subfields of the field $\mathbb{F}_{2^{35}}$ (finite field with 2^{35} elements) is

(a) 1. (b) 2. (c) 3. (d) 4. (e) none of these.

(15) Denote by $\mathbb{Z}_p$, the field of integers modulo p, where p is a prime. In the vector space $\mathbb{Z}_p$ the number of distinct one dimensional subspaces is

(a) $p+2$. (b) $p+1$. (c) $p-1$. (d) $p-2$. (e) none of these.

(16) If A is a 4×4 matrix with characteristic polynomial $(x^2 - 1)^2$ and minimal polynomial $(x^2 - 1)$, then its Jordan canonical form is

(a) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

(e) none of the above.

(17) Let X and Y be two topological spaces and $f : X \to Y$ be a continuous surjection. Then

(a) V is an open set in X implies $f(V)$ is open in Y.
(b) F is a closed set in X implies $f(F)$ is closed in Y.
(c) D is dense in X implies $f(D)$ is dense in Y.
(d) A is an infinite subset of X implies $f(A)$ is an infinite subset of Y.
(e) none of the above.

(18) Let m be the Lebesgue measure on $\mathbb{R}$ and E be a Lebesgue measurable subset of R. For any real a, if we denote $aE = \{ax : x \in E\}$, then

(a) $m(aE) = m(E)$.
(b) $m(aE) = am(E)$.
(c) $m(aE) = |a|m(E)$.
(d) $m(aE) = a + m(E)$.
(e) none of the above.

C-47

(19) On the vector space $C^1[0,1]$ of all real-valued continuously differentiable functions defined on $[0,1]$, consider the following norms $\|f\|_\infty = \sup_{0\le x\le 1}|f(x)|$, $\|f\|_1 = \int_0^1 |f(x)|dx$, $\|f\|_2 = \|f\|_\infty + \|f'\|_\infty$, and $\|f\|_3 = \max\{\|f\|_\infty, \|f'\|_\infty\}$. Then

(a) $\|\cdot\cdot\|_\infty$ and $\|\cdot\cdot\|_2$ are equivalent.
(b) $\|\cdot\cdot\|_1$ and $\|\cdot\cdot\|_2$ are equivalent.
(c) $\|\cdot\cdot\|_1$ and $\|\cdot\cdot\|_3$ are equivalent.
(d) $\|\cdot\cdot\|_\infty$ and $\|\cdot\cdot\|_3$ are equivalent.
(e) $\|\cdot\cdot\|_2$ and $\|\cdot\cdot\|_3$ are equivalent.

(20) Among the following statements, which one does not characterize a finite-dimensional normed linear space X?

(a) every linear subspace of X is closed.
(b) the closed unit ball $B = \{x \in X : \|x\| \le 1\}$ is compact.
(c) every bounded sequence in X has a convergent subsequence.
(d) every continuous functional on a linear subspace of X has a unique Hahn-Banach extension.
(e) every functional on X is continuous.

(21) Consider the linear subspace $Y = \{(x,y,z) \in \mathbb{R}^3 : x = y, z = 0\}$ of $\mathbb{R}^3$ with the norm $\|(x,y,z)\| = |x| + |y| + |z|$ and $f_0 : Y \to \mathbb{R}$ be a linear functional given $f_0(x,y,z) = 2x$, where $(x,y,z) \in Y$. Then

(a) $f_1 : \mathbb{R}^3 \to \mathbb{R}$ defined by $f_1(x,y,z) = 2x + z$ is a Hahn-Banach extension of f_0.
(b) $f_2 : \mathbb{R}^3 \to \mathbb{R}$ defined by $f_2(x,y,z) = 2y + z$ is a Hahn-Banach extension of f_0.
(c) $f_3 : \mathbb{R}^3 \to \mathbb{R}$ defined by $f_3(x,y,z) = x - y$ is a Hahn-Banach extension of f_0.
(d) $f_4 : \mathbb{R}^3 \to \mathbb{R}$ defined by $f_4(x,y,z) = x + y - z$ is a Hahn-Banach extension of f_0.
(e) none of the above is a Hahn-Banach extension of f_0.

(22) The general solution of the fourth order differential equation $y^{(iv)} - 5y''' + 6y'' + 4y' - 8y = 0$ is

(a) $y = (c_1 + c_2x + c_3x^2)e^{2x} + c_4e^{-x}$.
(b) $y = c_1e^{2x} + c_2e^{-x}$.
(c) $y = c_1 + c_2xe^{x} + c_3x^2e^{2x} + c_4e^{-x}$.
(d) $y = (c_1 + c_2x + c_3x^2)e^{-x} + c_4e^{2x}$.
(e) none of the above.

(23) The critical point $(0, 0)$ of the following system

$$\frac{dx}{dt} = (y+1)^2 - \cos x,$$
$$\frac{dy}{dt} = \sin(x+y),$$

is

(a) a saddle point.
(b) an unstable node.
(c) a stable node.
(d) an unstable spiral.
(e) a stable spiral.

(24) The extremum of the functional $J[y] = \int_0^1 (y^2 + 2yy')dx$, satisfying $y(0) = 1$ and $y(1) = 0$

(a) is a straight line
(b) is a parabola.
(c) is an ellipse
(d) does not exist.
(e) none of the above.

(25) The equation $u_{xx} + xu_{xy} + yu_{yy} - xyu_x = 0$ is hyperbolic for

(a) $y > \frac{x^2}{4}$.
(b) $y < \frac{x^2}{4}$.
(c) $y = \frac{x^2}{4}$.
(d) $y = x$.
(e) none of the above.

Part B (50 Marks)

There are 15 questions in this part. Each question carries 5 marks. Attempt **any 10 questions**. The answers should be written in the separate booklet given.

(1) Consider the sequence $\{f_n\}$ of real valued functions, where $f_n(x) = \frac{1}{1+nx}$, $x \in \mathbb{R}$. Answer the following questions with proper justification.

(a) Is $\{f_n\}$ pointwise convergent on $[0, 1]$?
(b) Is $\{f_n\}$ uniformly convergent on $[0, 1]$?
(c) Is $\{f_n\}$ uniformly bounded on $[0, 1]$?
(d) Is $\{f_n\}$ bounded in $L^1[0, 1]$?
(e) Is $\{f_n\}$ convergent on $L^1[0, 1]$?

C-47

(2) If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then show that f is either continuous at each point of $\mathbb{R}$ or not continuous at any point of $\mathbb{R}$.

(3) Let $a > 0$ and $\{x_n\}$ be a sequence such that $x_1 = 1, x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$ for $n = 1, 2, \ldots$. Show that $\lim_{n\to\infty} x_n = \sqrt{a}$.

(4) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = x^2y + e^x + z$ and $\underset{\sim}{a} = (0, 1, -1)$. Using implicit function theorem, find the implicit function $g(y, z)$ in a neighbourhood of $(1, -1)$ in $\mathbb{R}^2$ and compute $\frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial z}$ at $(1, -1)$.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f has derivatives of all orders on $\mathbb{R}$ and $f^k(0) = 0$ for all integers $k \geq 1$.

(6) Determine the number of roots of the polynomial $3z^7 + 5z - 1$ inside the circles $|z| = 1, |z| = 2$ and in the annulus $1 < |z| < 2$.

(7) Using Residue theorem, find the value of the integral

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx.$$

(8) Let M_n^λ stand for the $n \times n$ matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 & . & 0 & 0 \\ 0 & \lambda & 1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & . & 0 & \lambda \end{pmatrix}.$$

What will be the characteristic polynomial and the minimal polynomial of the 9×9 matrix whose diagonal blocks are $M_3^2, M_2^2, M_1^2, M_2^{-2}, M_1^{-2}$. Justify your answer.

(9) Let T be a non-zero $n \times n$ matrix. If T commutes with all $n \times n$ matrices, then show that T is a scalar multiple of the identity matrix.

(10) Show that a group of order 30 must contain a subgroup of order 15. Is it necessarily normal?

(11) Let α be a root of $x^3 + x + 1$ and β be a root of $x^3 + x + 3$. Show that it is not possible that $\alpha \in \mathbb{Q}(\beta)$.

(12) Suppose the density of a subset A of $\mathbb{N}$, the set of natural numbers is defined as $\lim_{n\to\infty} \frac{a_n}{n}$, if it exists, where a_n is the cardinality of the set $A \cap \{1, 2, \ldots, n\}$. The density may or may not exist for a subset A of $\mathbb{N}$. Clearly if it exists, then it is a number in $[0, 1]$.

C-47

(a) If m is a fixed positive integer and $A = \{km : k \in \mathbb{N}\}$, then find the density of A.

(b) What is the density of the set $\{2^n : n \in \mathbb{N}\}$?

(13) Consider the Banach spaces $C[0,1]$ of all real-valued continuous functions on $[0,1]$ with the supremum norm $\|f\|_\infty$ and $C^1[0,1]$ of all real valued continuously differentiable functions on $[0,1]$ with the norm $\|f\|_\infty + \|f'\|_\infty$. For $f \in C[0,1]$, define $\tilde{f}$ on $[0,1]$ by $\tilde{f}(x) = \int_0^x f(t)dt$, $0 \le x \le 1$, which belongs to $C^1[0,1]$. Define $T : C[0,1] \to C^1[0,1]$ by $Tf = \tilde{f}$. Show that T is a bounded linear map. Find the value of $\|T\|$.

(14) Find the characteristic values and characteristic functions of

$$\frac{d}{dx}(x\frac{dy}{dx}) + \frac{\lambda}{x}y = 0, \ y(1) = 0, \ y(e^\pi) = 0,$$

where λ is a non-negative number.

(15) Find the integral surface for the partial differential equation $z(xp - yq) = y^2 - x^2$, passing through the initial data curve $(2s, s, s)$.