AA-3353

Seat No. _______

M. Phil. Examination

April / May – 2003

Mathematics : Paper - I

Time : Hours] [Total Marks : **75**

1. (a) Let G be a finite abelian group of order n and m be a positive integer dividing n. Show that G has a subgroup of order m. (10)
   (b) Show that every infinite group has infinitely many distinct subgroups. (5)

OR

1. (a) Are any two of additive groups **Z, Q, R** isomorphic? Explain. (5)
   (b) Give an example of an infinite group G such that each $x \in G, x \neq e$ has the same finite order. (5)
   (c) Can a group G have two distinct subgroups H_1 and H_2 of order 5 such that $H_1 \cap H_2 \neq \{e\}$? (5)

2. (a) Let G be a finite abelian group and (n,O(G))=1. Prove that every $g \in G$ can be written as $g = x^n$ for some $x \in G$. (8)
   (b) Identify all homomorphisms of the ring **Z** to itself. (7)

OR

2. (a) Let F be a field. Define the operation * on F by a*b = a + b –ab, a,b $\in F$. Prove that $\{x \in F / x \neq 1\}$ forms a group under *, which is isomorphic to the multiplicative group $\{x \in F / x \neq 0\}$. (10)
   (b) Show that a commutative ring D is an integral domain iff for a, b, c in D, with $a \neq 0$ the relation ab=ac implies b=c. (5)

3. (a) For $n \geq 1$ let $x_n = \dfrac{1.3.5....(2n-1)}{2.4.6.........(2n)}$ and $y_n = \dfrac{1}{n^2 x_n}$. Does $\{x_n\}$ converge? Does $\{y_n\}$ converge? Justify. (9)
   (b) Let $f : R \to R$ be continuous with $f'(x) = 0$ for each $x \neq 0$. Show that f is constant. (6)

OR

3. (a) Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$, $n \geq 1$. Show that $\{x_n\}$ converges and $\lim_n x_n = 2$. (6)
   (b) Let $f : R \to R$ and $g : R \to R$ be continuous functions. Show that h=max{f,g} is continuous on R. (5)
   (c) If $f : [a,b] \to R$ is continuous, show that there is $g : [a,b] \to R$ such that $g'(x) = f(x), \forall x \in [a,b]$. (4)

Q.4. Attempt any three of the following: (15)

(a) Show that the space X is disconnected iff there exists a continuous function $f: X \to \{0,1\}$ which is onto.

(b) Show that a convex subset of a normed linear space is connected. Can you say something more? Justify.

(c) Suppose X is a Hausdorff space and $f: R \to X$ is a continuous function such that f(x)=x for all rational x. Show that f(x)=x for all real x. Can you drop the condition of Hausdorffness? Justify.

(d) Show that a uniformly continuous image of a Cauchy sequence is Cauchy. Is this true for a continuous image? Justify.

(e) " A subset of a metric space is compact iff it is closed and bounded." Is the above true both-way, one-way or no-way? Justify your answer.

Q.5. Attempt any three of the following: (15)

(a) Give a subset X of [0,1] for which $\overline{X} \setminus X$ is infinite but the subset X is discrete as a subspace of [0,1].

(b) "$x \in \overline{A}$ iff there exists a sequence $<x_n>$ in A converging to x" Which implication is always true? Which one is false? Justify.

(c) Define a complete metric on (0,1) and a non-complete metric on R.

(d) Are the following true? (i) $\overline{A}' = \text{Int}(A')$ (ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Justify.

(e) Show that P the set of irrationals is not locally compact.
