MAY 2011 **P/ID 4524/XDA**

Time : Three hours Maximum : 100 marks

SECTION A — ($4 \times 20 = 80$ marks)

Answer ALL questions.

Each question carries 20 marks.

1. (a) (i) State and prove second part of Sylow's theorem.

   (ii) Prove that $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.

Or

   (b) (i) State and prove the class equation of G.

   (ii) Prove that any finitely generated module over an Euclidean ring is the direct sum of finite number of cyclic submodules.

2. (a) (i) If $p(x)$ is a polynomial in $F(x)$ of degree $n \geq 1$ and is irreducible over F, then show that there is an extension E of F such that [E:F]=n, in which $p(x)$ has a root.

(ii) If K is a finite extension of F then show that $G(K,F)$ is a finite group and its order, $O(G(K,F))$ satisfies $O(G(K,F)) \leq [K : F]$.

Or

(b) (i) Prove that K is a normal extension of F if and only if K is the splitting field of some polynomial over F.

(ii) Show that a polynomial of degree n over a field can have at most n roots in any extension field.

3. (a) (i) Show that G is solvable if and only if $G^{(k)} = e$ for some integer k.

(ii) Show that for every prime number p and every positive integer m there is a unique field having p^m elements.

Or

(b) (i) If $T \in A(V)$ has all its characteristic roots in F then show that there is a basis of V in which the matrix of T is triangular.

(ii) State and prove Wedderburn's theorem.

4. (a) (i) Show that the elements S and T in $A_F(V)$ are similar in $A_F(V)$ if and only if they have the same elementary divisors.

(ii) Determine the rank and signature of the quadratic form

$$x_1^2 + x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + 2x_3^2.$$

Or

(b) (i) For each $i = 1,2,\ldots,k,\ V_i \neq (0)$ and V=$V_1 \oplus V_2 \oplus \ldots. \oplus V_R$, show that the minimal polynomial of T_i is $q_i(x)^{l_i}$.

(ii) If F is a field of characteristic 0, and if $T \in A_F(V)$ is such that $trT^i = 0$ for all $i \geq 1$ then show that T is nilpotent.

SECTION B — (10 × 2 = 20 marks)

Answer any TEN questions.

Each question carries 2 marks.

5. List all conjugate classes in S_3.

6. Find the number of 11-Sylow subgroups in a group of order 231.

7. Give an example of a module over the ring of integers.

8. What is the degree of $\sqrt{2} + \sqrt{3}$ over Q?

9. Determine the degree of splitting field of the polynomial $x^4 - 2$.

10. Define a perfect field.

11. Complete G (K, F) where K is the field of complex numbers and F is the field of real numbers.

12. Define the Galois group of $f(x)$.

13. If G is a solvable group and if $\overline{G}$ is a homomorphic image of G then show that $\overline{G}$ is solvable.

14. If M, of dimension m, is cyclic with respect to T, then show that the dimension of MT^k is $m - k$ for all $k \leq m$.

15. Let F be a finite field. Then show that F has p^m elements where the prime number p is the characteristic of F.

16. Define a normal extension.

17. If $T \in A(V)$ then show that $\text{tr}\,\text{T}$ is the sum of the characteristic roots of T.w.

18. Define Hermitian adjoint of a linear transformation.

19. If $T \in A(V)$ is Hermitian then show that all its characterstic roots are real.

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