2009

BOOKLET NO. TEST CODE : **SIB**

Afternoon

Questions : 10	**Time : 2 hours**

- *Write your Name, Registration Number, Test Code and the Number of this Booklet in the appropriate places on the answer booklet.*
- All questions carry equal marks. Answer as many as you can.
- Give complete answers showing clearly how the steps are obtained. However, you may get some credit for partially correct answers.
- All rough work must be done on the answer booklet.
- You are not allowed to use calculators in any form.

WAIT FOR THE SIGNAL TO START.

Answer to each question should start on a fresh page.
Indicate the correct question numbers against your answers.

1. Two train lines intersect each other at a junction at an acute angle θ. A train is passing along one of the two lines. When the front of the train is at the junction, the train subtends an angle α at a station on the other line. It subtends an angle β $(< \alpha)$ at the same station, when its rear is at the junction. Show that
$$\tan\theta = \frac{2\sin\alpha\sin\beta}{\sin(\alpha-\beta)}.$$

2. Let $f(x)$ be a continuous function, whose first and second derivatives are continuous on $[0, 2\pi]$ and $f''(x) \geq 0$ for all x in $[0, 2\pi]$. Show that
$$\int_0^{2\pi} f(x)\cos x\, dx \geq 0.$$

3. Let ABC be a right-angled triangle with $BC = AC = 1$. Let P be any point on AB. Draw perpendiculars PQ and PR on AC and BC respectively from P. Define M to be the maximum of the areas of BPR, APQ and $PQCR$. Find the minimum possible value of M.

4. A sequence is called an *arithmetic progression of the first order* if the differences of the successive terms are constant. It is called an *arithmetic progression of the second order* if the differences of the successive terms form an arithmetic progression of the first order. In general, for $k \geq 2$, a sequence is called an *arithmetic progression of the k-th order* if the differences of the successive terms form an arithmetic progression of the $(k-1)$-th order.

   The numbers
$$4, 6, 13, 27, 50, 84$$
   are the first six terms of an arithmetic progression of some order. What is its least possible order? Find a formula for the n-th term of this progression.

5. A cardboard box in the shape of a rectangular parallelopiped is to be enclosed in a cylindrical container with a hemispherical lid. If the total height of the container from the base to the top of the lid is 60 centimetres and its base has radius 30 centimetres, find the volume of the largest box that can be completely enclosed inside the container with the lid on.

6. Let $f(x)$ be the function satisfying

$$xf(x) = \log x, \qquad \text{for } x > 0.$$

Show that $f^{(n)}(1) = (-1)^{n+1}n!\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$, where $f^{(n)}(x)$ denotes the n-th derivative of the function f evaluated at x.

7. Show that the vertices of a regular pentagon are concyclic. If the length of each side of the pentagon is x, show that the radius of the circumcircle is $\frac{x}{2}\operatorname{cosec} 36^\circ$.

8. Find the number of ways in which three numbers can be selected from the set $\{1, 2, \ldots, 4n\}$, such that the sum of the three selected numbers is divisible by 4.

9. Consider 6 points located at $P_0 = (0, 0)$, $P_1 = (0, 4)$, $P_2 = (4, 0)$, $P_3 = (-2, -2)$, $P_4 = (3, 3)$ and $P_5 = (5, 5)$. Let R be the region consisting of *all* points in the plane whose distance from P_0 is smaller than that from any other P_i, $i = 1, 2, 3, 4, 5$. Find the perimeter of the region R.

10. Let x_n be the n-th non-square positive integer. Thus, $x_1 = 2$, $x_2 = 3$, $x_3 = 5$, $x_4 = 6$, etc. For a positive real number x, denote the integer closest to it by $\langle x \rangle$. If $x = m + 0.5$, where m is an integer, then define $\langle x \rangle = m$. For example, $\langle 1.2 \rangle = 1$, $\langle 2.8 \rangle = 3$, $\langle 3.5 \rangle = 3$. Show that $x_n = n + \langle \sqrt{n} \rangle$.