A great part of its [*higher arithmetic*] theories derives an additional charm from the peculiarity that important propositions, with the impress of simplicity on them, are often easily discovered by induction, and yet are of so profound a character that we cannot find the demonstrations till after many vain attempts; and even then when we do succeed, it is often by some tedious and artificial process, while the simple methods may long remain concealed.

— Carl Friedrich Gauss

CONCEPT OF SELF-INDUCTION

by Mohamad Asif

Whenever the electric current flowing through a circuit (*or loop*) changes, the magnetic flux linked with the circuit (*or loop*) also changes so as to produce an induced emf current in the circuit (*or loop*). This phenomenon is called the phenomenon of self-induction and the emf-induced is called back-emf.

Circuit element that has a large self-inductance is called an inductor. Its symbol is .

Definition

Self-induction is the property of a coil by virtue of which it opposes the growth (*or*) decay of the current flowing through it.

- Self-induction is also known as inertia of electricity as it opposes the growth (*or*) decay of the current in circuit.
- Induction may be viewed as electrical inertia. It is analogous to inertia in mechanics. It does not oppose the current, but it opposes the change in current.

Coefficient of Self-Induction (or) Self-Inductance [L]

The magnetic flux (ϕ) linked with the coil is found to be proportional to the strength of the current (i)

i.e., $\phi \propto i \Rightarrow \phi = Li$

where, L = coefficient of self-induction

If $i = 1\text{A} \Rightarrow \phi = i$

Definition of [L]

It is defined as the magnetic flux linked with the coil when unit current flows in it.

According to Faraday's law of electromagnetic induction induced emf in the coil is given by

$$e = -N\frac{d\phi}{dt}$$

$$\Rightarrow \quad e = -\frac{d(Li)}{dt} = -L\frac{di}{dt}$$

If $\dfrac{di}{dt} = 1$

i.e., if rate of change is unity, we get $L = -e$

Hence, L can also be defined as 'coefficient of self-induction of a coil is defined as negative induced emf in the coil when the rate of change of current in the coil is unity'.

SI unit of L is Henry (H); other units are Weber/Ampere, volt-second/ampere, J/A^2.

Wb^2/J, $\text{V-s}^2\ \text{C}^{-1}$

Self-inductance of a circular coil is $B = \dfrac{\mu_0 Ni}{2r}$.

$$\therefore \quad \phi = NBA = N\left(\frac{\mu_0 Ni}{2r}\right)\pi r^2 = \frac{\mu_0 \pi N^2 ri}{2}$$

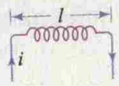
Now, comparing with $N\phi_B = Li$, we get $L = \dfrac{\mu_0 \pi N^2 r}{2}$

Self-Inductance of a Solenoid

Consider a long solenoid of length l, area of cross-section A and number of turns per unit length n and length is very large when compared with radius of cross-section.

For solenoid, $B = \mu_0 ni$.

Total number of turns in the solenoid of length l, $N = nl$.



Now, the magnetic flux linked with each turn of the solenoid $= B \times A = \mu_0 niA$.

$\therefore$ Total magnetic flux linked with the whole solenoid.

ϕ = magnetic flux with each turn × number of turns in the solenoid

$$\phi = \mu_0 niA \times nl = \mu_0 n^2 iAl$$

But, $\phi = Li$

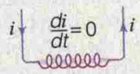
$$\therefore \quad Li = \mu_0 n^2 iAl$$

$$L = \mu_0 n^2 Al$$

Since, $n = \dfrac{N}{l} \Rightarrow L = \mu_0 \dfrac{N^2}{l}\cdot A$

1. An ideal inductor has inductance and no resistance.
2. One can have resistance without inductance.
3. More is the permeability of medium, more is the self-inductance.
4. An inductor will have large inductance and low resistance.
5. The direction of induced emf for different states of current in a coil.

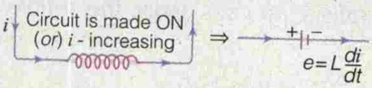
(a) Steady current



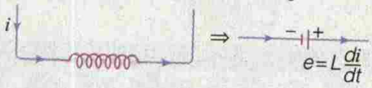
$e = 0$

No opposition.

(b) Make of circuit (*or*) increasing current



(c) Breaking of circuit (*or*) decreasing of current



Energy Stored in an Inductor

The work done in maintaining the current through the inductor is stored as the potential energy in its magnetic field

$$U = \frac{1}{2} Li^2$$

(a) When $i = 1A$

$$\Rightarrow \quad L = 2U$$

(b) Induced power $P = e \times i = Li\left(\frac{di}{dt}\right)$.

(c) In case of solenoid $L = \mu_0 n^2 Al$.

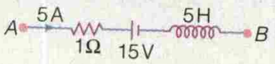
$\therefore$ Magnetic energy stored per unit volume

$$U_B = \frac{\frac{1}{2} Li^2}{Al} \Rightarrow U_B = \frac{1}{2}\mu_0 n^2 i^2$$

Hence, $$U_B = \frac{B^2}{2\mu_0}$$

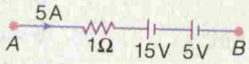
Question *The network shown is a part of the closed circuit in which the current is changing. At an instant, current in it is 5 A. Find the potential difference between the points A and B when the current is increasing at 1 A/s.*

Solution The coil can be imagined as a cell of emf



$$e = L\left(\frac{di}{dt}\right) = 5 \times 1 = 5V$$

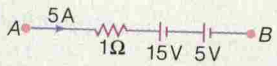
$\therefore$ Equivalent circuit is



$$V_A - 5(1) - 15 - 5 = V_B$$

Hence, $V_A - V_B = 5 + 15 + 5 = 25$ V

If current is decreasing at 1A/s



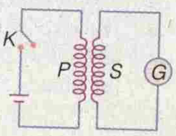
$$V_A - 5(1) - 15 + 5 = V_B$$

$$V_A - V_B = 5 + 15 - 5 = 15 \text{ V}$$

You don't have to be good at everything. But you have got to be the best in something.

Mutual-Induction

The phenomenon of production of emf in a coil when the current in neighbouring coil changes is called 'mutual induction'. The circuit in which the current changes is called the primary circuit, while the neighbouring circuit in which emf is induced, is called the secondary circuit.



The direction is given by Lenz's law.

Coefficient of mutual-induction

$$\phi_s \propto i_p$$
$$\phi_s = Mi_p$$

where M = coefficient of mutual induction

According to Faraday's law of electromagnetic induction

$$e = -\frac{d\phi}{dt} \quad \therefore \quad e_s = -\frac{d\phi_s}{dt}$$

we get, $$e_s = -\frac{d(Mi_p)}{dt} = -M\frac{di_p}{dt}$$

If $\frac{-di_p}{dt} = 1$ then, $M = e$

- One cannot have mutual-inductance without self-inductance.
- Mutual-inductance depends on orientation of the coils (*i.e.*) angle between the axis of the coils.
- If the axes are parallel, then M is maximum.
- If axes are perpendicular then M is minimum.

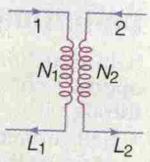
Inductors in series and parallel

$$L_s = L_1 + L_2 + L_3 + \ldots + L_n$$

$$\frac{1}{L_p} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \ldots + \frac{1}{L_n}$$

Relation between L_1, L_2 and M

The flux linked with coil 1 is



$$N_1\phi_1 = L_1 i_1$$

$$L_1 = \frac{N_1\phi_1}{i_1}$$

The flux linked with coil 2 is $N_2\phi_2 = L_2 i_2$

$$L_2 = \frac{N_2\phi_2}{i_2}$$

M on 1 because of 2, $$M_{12} = \frac{N_1\phi_1}{i_2}$$

M on 2 because of 1, $$M_{21} = \frac{N_2\phi_2}{i_1}$$

If the flux in linkage is maximum, then

$$M_{12} = M_{21} = M$$

$$M_{12} \times M_{21} = \frac{N_2\phi_2}{i_1} \times \frac{N_1\phi_1}{i_2}$$

$$M^2 = L_1 L_2$$

$$\therefore \quad M = \sqrt{L_1 L_2}$$

This is the maximum mutual-inductance when all the flux linked with one coil is also completely linked with the other. In general, only a fraction of the total flux will be linked with the coil due to flux leakage.

$$\therefore \quad M = k\sqrt{L_1 L_2}$$

where, k = coefficient of coupling ($k \le 1$)

For tight coupling (*or*) if the coils are closely wound then $k = 1$

$$\therefore \quad M_{max} = \sqrt{L_1 L_2}$$

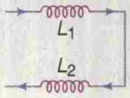
Let two coils of inductances L_1 and L_2 are connected in series and M is their mutual-inductance.

(a) If the two coils support each other, then

$$L = (L_1 + M) + (L_2 + M)$$
$$= L_1 + L_2 + 2M$$

(b) If the two coils oppose each other, then

$$L = (L_1 - M) + (L_2 - M)$$
$$= L_1 + L_2 - 2M$$



Special Case

If two coils of self-inductances L_1 and L_2, having mutual-inductance M are connected in parallel and placed close to each other, then the net inductances is given by

$$L_p = L_{\text{parallel}} = \frac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M}$$

Solved Examples

Example 1 *A small coil of radius r is placed at the centre of a large coil of radius R, where R>> r. The coils are coplanar. The mutual inductance between the coils is*

(a) $\frac{\mu_0 \pi r}{2R}$ (b) $\frac{\mu_0 \pi r^2}{2R}$ (c) $\frac{\mu_0 \pi r^2}{2R^2}$ (d) $\frac{\mu_0 \pi r}{2R^2}$

Solution. (b) $B = \frac{\mu_0 i}{2R}$, ϕ = (Area of smaller coil) B

$$\phi = \pi r^2 \times B$$

$$\phi = \pi r^2 \times \frac{\mu_0 i}{2R}$$

$$M = \frac{\phi}{i} = \frac{\mu_0 \pi r^2}{2R}$$

Example 2 *A self-induced emf in a solenoid of inductance L changes in time as $\varepsilon = \varepsilon_0 e^{-kt}$. Find the total charge that passes through the solenoid assuming the charge is finite.*

Solution. Since, $\varepsilon = \varepsilon_0 e^{-kt}$ and $\varepsilon = -L\frac{di}{dt} \Rightarrow -L \cdot \frac{di}{dt} = \varepsilon_0 e^{-kt}$

Since, we are to find the current as a function of time (t), so let the current be i at time t. Then as $t \to \infty, I \to 0$ [$\because \varepsilon \to 0$]

$$\Rightarrow \quad \int_i^0 di = -\frac{\varepsilon_0}{L}\int_t^\infty e^{-kt} dt$$

$$-i = \frac{\varepsilon_0}{kL} e^{-kt}\Big|_t^\infty = \frac{\varepsilon_0}{kL}[e^{-\infty} - e^{-kt}]$$

$$i = \frac{\varepsilon_0}{kL} e^{-kt}$$

As $$Q = \int_0^\infty i\,dt = \int_t^\infty \frac{\varepsilon_0}{kL} e^{-kt} \cdot dt = \frac{\varepsilon_0}{kL}\int_0^\infty e^{-kt} \cdot dt = \frac{\varepsilon_0}{k^2 L}$$

$$\therefore \quad Q = \frac{\varepsilon_0}{k^2 L}$$

Example 3 *Assume that the magnitude of the magnetic field outside a sphere of radius R is $B = B_0\left(\frac{R}{r}\right)^2$. Where B_0 is a constant. Determine the total energy stored in the magnetic field outside the sphere.*

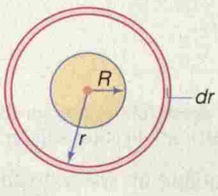
Solution. Since, $U_m = \frac{B^2}{2\mu_0} = \left(\frac{B_0^2}{2\mu_0}\right)\left(\frac{R}{r}\right)^4$

The magnetic energy is a spherical shell of radius r thickness dr is given by

$$dU = U \cdot dV$$

Where $dV = 4\pi r^2 \cdot dr$

$$dU = \left(\frac{B_0^2 R^4}{2\mu_0}\right)\left(\frac{4\pi r^2 \cdot dr}{r^4}\right)$$



$$\left[\because \int_R^\infty r^{-2} dr = \frac{1}{R}\right]$$

$$U = \int dU = \frac{2\pi B_0^2 R^4}{\mu_0}\int_R^\infty r^{-2} dr = \frac{2\pi B_0^2 R^3}{\mu_0}$$

Example 4 *A large coil of radius R_1 and having N_1 turns is coaxial with a small coil radius R_2 and having N_2 turns. The centres of the coils are separated by a distance x that is much larger than R_1 and R_2. What is the mutual-inductance of the coils?*

Solution. $B_1 = \frac{\mu_0 N_1 i_1 R_1^2}{2(x^2 + R_1^2)^{3/2}}$

This field B_1 is normal to area of coil 2 and is nearly uniform over this area. So, it produces a flux.

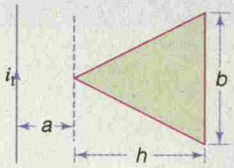
$$\phi_{12} = N_2 B_1 A_2 = \frac{\mu_0 N_1 N_2 i_1 R_1^2 (\pi R_2^2)}{2(x^2 + R_1^2)^{3/2}}$$

When i_1 varies, then

$$\varepsilon_2 = -\frac{d\phi_{12}}{dt} = -\frac{\mu_0 \pi (N_1 N_2)(R_1 R_2)^2}{2(x^2 + R_1^2)^{3/2}} \cdot \frac{di_1}{dt} = -M \cdot \frac{di}{dt}$$

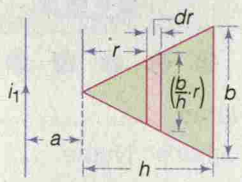
$$\Rightarrow \quad M = \frac{\mu_0 \pi (N_1 N_2)(R_1 R_2)^2}{2(x^2 + R_1^2)^{3/2}}$$

Example 5 *Figure shows a long wire and a triangular coil. Calculate the mutual-inductance of the combination.*



Solution. Consider a differential strip of area dA then from property of similar triangles length of the strip is $\frac{br}{h}$.

$$dA = \left(\frac{b}{h}r\right)dr$$



Let i_1 be the current in wire, then its magnetic field on the strip is

$$B = \frac{\mu_0 i}{2\pi(a + r)}$$

If $$d\phi_{B_{12}} = B_1 dA = \frac{\mu_0 i_1 br \cdot dr}{2\pi h(a + r)}$$

$$\phi_{B_{12}} = \frac{\mu_0 ib}{2\pi h}\int_0^h \frac{(a + r) - a}{(a + r)} \cdot dr = \frac{\mu_0 ib}{2\pi h}\int_0^h \left(1 - \frac{a}{a + r}\right) \cdot dr$$

$$\phi_{B_{12}} = \frac{\mu_0 ib}{2\pi h}\left[h - a\log_e\left(\frac{a + h}{2}\right)\right]$$

$$\therefore \quad M_{12} = \frac{\phi_{B_{12}}}{I} = \frac{\mu_0 b}{2\pi h}\left[h - a\log_e\left(\frac{a + h}{a}\right)\right]$$

Growth and Decay of Current in L-R Circuit

(a) **Growth of current in L-R circuit**

$$i = i_0\left(1 - e^{\frac{-Rt}{L}}\right)$$

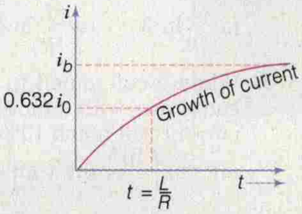
when $t = \frac{L}{R}$

$$i = i_0(1 - e^{-1})$$

$$= i_0\left(1 - \frac{1}{e}\right)$$

$$i = i_0\left(1 - \frac{1}{2.718}\right) = 0.632\, i_0$$

and $\frac{i}{i_0} \times 100 = 63.2\%$



The time in which the current grows from zero to 0.632 times (*or*) 63.2% of its maximum value, is called inductive-time constant.

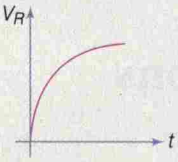
- At $t = 0$, inductor offers infinite resistance.
- At $t = \infty$, inductor offers zero resistance.
- The rate of growth of current is given by

$$\frac{di}{dt} = -\frac{Ri_0}{L} e^{-\frac{Rt}{L}}$$

Potential difference across the resistor is

$$V_R = iR = i_0 R(1 - e^{\frac{-Rt}{L}})$$

$$V_R = E(1 - e^{-\frac{Rt}{L}})$$



Potential difference across the inductor

$$V_L = E - V_R$$

$$V_L = E - E[1 - e^{-\frac{R}{L}t}]$$

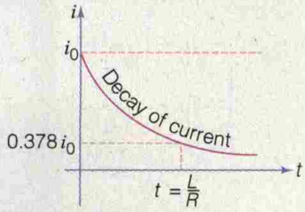
$$V_L = E\, e^{-\frac{R}{L}t}$$

(b) **Decay of current in *L-R* circuit**

$$i = i_0 e^{-\frac{Rt}{L}}$$

when $t = \frac{L}{R}$,

$$i = i_0 e^{\frac{-RL}{LR}} = i_0 e^{-1}$$

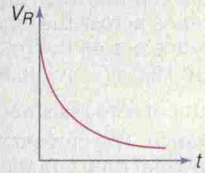


$$i = \frac{i_0}{e} = \frac{i_0}{2.718} = 0.368\, i_0$$

Rate of decay of current is given by

$$\frac{di}{dt} = -\frac{i_0 R}{L} e^{-\frac{Rt}{L}}$$

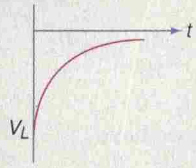
(i) Potential difference across resistor is



$$V_R = iR = i_0 R e^{-\frac{Rt}{L}}$$

$$V_R = E_0 e^{-\frac{R}{L}t}$$

(ii) Potential difference across inductor is



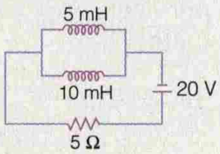
$$V_L = -V_R = E_0 e^{-\frac{R}{L}t}$$

"You're like a Aladdin and universe is like a Genie. When you really want something wish for it. Sincerely from your heart and start trying for it. The entire universe will re-arrange itself and every element of the universe will help you realizing your dream..."

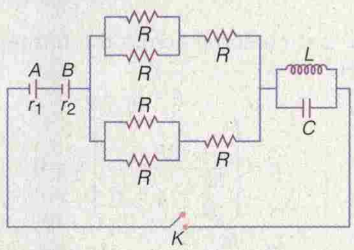
Strategy without tactics is the slowest route to victory. Tactics without strategy is just the noise before defeat.

'Smart' Practice

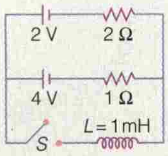
1. In the given circuit, find the current through 5 mH inductor in steady state.



2. In the circuit shown, *A* and *B* are two cells of same emf *E* but different internal resistances r_1 and r_2 $(r_1 > r_2)$, respectively. Find the value of *R* such that the potential difference across the terminals of cell *A* is zero a long time after the key *K* is closed.



3. In the circuit shown, switch S is closed at time $t = 0$. Find the current through the inductor as a function of time t.



4. A coil of inductance 8.4 mH and resistance 6 Ω are connected to a 12 V battery. At what time the current in the coil will be 1 A?

(a) 1×10^{-3} s (b) 1×10^{-2} s
(c) 1×10^{-6} s (d) 1×10^{-4} s

5. A 4 μF capacitor and a resistance 2.5 MΩ are in series with 12 V battery. Find the time after which the potential difference across the capacitor is 3 times the potential difference across the resistor? (ln 2 = 0.693)

(a) 13.86 s (b) 12.2 s (c) 8.36 s (d) 16.86 s

6. At $t = 0$ an inductor of zero resistance is joined to a cell E through a resistance. The current increases with a time constant τ. After what time will the potential difference across the coil be equal to that across the resistance?

(a) $\frac{L}{R}\ln 3$ (b) $\frac{L}{R}\ln 4$ (c) $\frac{L}{R}\ln 2$ (d) $\frac{L}{R}\ln 6$

7. When a coil joined to a cell, the current through the coil grows with a time constant τ. After what time the current will reach 10% of its steady-state value?

(a) $\tau \ln\left(\frac{10}{9}\right)$ (b) $\tau \ln\left(\frac{9}{10}\right)$ (c) $\tau \ln\left(\frac{6}{19}\right)$ (d) $\tau \ln\left(\frac{3}{2}\right)$

8. The time constant of a certain inductive coil was found to be 2.5 ms with a resistance of 80 Ω added in series, a new time constant of 0.5 ms was obtained. Find the inductance and resistance of the coil.

(a) 50 mH (b) 60 mH (c) 40 mH (d) 20 mH

9. Calculate the back emf of 10 H, 200 Ω coil 100 ms after a 100 V DC supply is connected to it.

(a) 113.5 V (b) 12.2 V (c) 213.6 V (d) 13.5 V

10. A coil having resistance 15 Ω and inductance 10 H connected across a 90 V DC supply. Determine the value of current after 2 s. What is the energy stored in the magnetic field at that instant?

(a) 162.45 J (b) 121.65 J (c) 24.2 J (d) 36.2 J

11. A cell of 1.5 V is connected across an inductor of 2 mH in series with a 2 Ω resistor. What is the rate of growth of current immediately after the coil is switched on

(a) 750 As^{-1} (b) 620 As^{-1} (c) 220 As^{-1} (d) 320 As^{-1}

Answers with Explanations

1. $20 - (i_1 + i_2) R = 5\frac{di_1}{dt} = 10\frac{di_2}{dt}$

At $t = 0, i_1 = i_2 = 0$

In steady state, we have

$$\frac{di_1}{dt} = \frac{di_2}{dt} = 0 \qquad (\because i_1 = 2i_2)$$

$$i_1 + i_2 = \frac{20}{R} = \frac{20}{5} = 4\text{A}$$

$$i_1 + \frac{i_1}{2} = 4\text{A}$$

$$3i_1 = 8$$

$$i_1 = \frac{8}{3}\text{A}$$

2. After long time, resistance across on inductor be comes zero while resistance across capacitor becomes infinite.

Hence, net external resistance $R_{\text{net}} = \dfrac{\frac{R}{2} + R}{2} = \dfrac{3R}{4}$

Current through batteries, $i = \dfrac{2E}{\frac{3R}{4} + r_1 + r_2}$

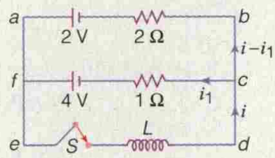
Given that potential across the terminals of cell A is zero, so

$$E - ir_1 = 0$$

$$E - \left(\frac{2E}{\frac{3R}{4} + r_1 + r_2}\right) r_1 = 0$$

$$R = \frac{4}{3}(r_1 - r_2)$$

3. For loop $abcfa$



$$-2 + 2(i - i_1) - i_1 + 4 = 0$$

$$-3i_1 + 2i = -2 \qquad \ldots\text{(i)}$$

For loop $fcdef$, we get

$$-4 + i_1 + L\frac{di}{dt} = 0$$

$$i_1 + L\frac{di}{dt} = 4 \qquad \ldots\text{(ii)}$$

Multiply Eq. (ii) by 3 and adding to Eq. (i), we get

$$2i_1 + 3L\frac{di}{dt} = 10$$

where, $L = 10^{-3}$H

$$\Rightarrow \quad 3L\frac{di}{dt} = 10 - 2i$$

$$\int_0^i \frac{di}{10 - 2i} = \int_0^t \frac{dt}{3L}$$

$$-\frac{1}{2}\log_e\left(\frac{10 - 2i}{10}\right) = \frac{t}{3L}$$

$$\left(\frac{10 - 2i}{10}\right) = e^{-\frac{2t}{3L}}$$

$$10 - 2i = 10\,e^{-\frac{2t}{3L}}$$

$$i = 5\left(1 - e^{-\frac{2000t}{3}}\right)\text{A}$$

4. (a) **5.** (a) **6.** (c) **7.** (a) **8.** (a)
9. (d) **10.** (a) **11.** (a)