## TEST CODE: MIII (Objective type) 2008

## SYLLABUS

**Algebra** — Permutations and combinations. Binomial theorem. Theory of equations. Inequalities. Complex numbers and De Moivre's theorem. Elementary set theory. Simple properties of a group. Functions and relations. Algebra of matrices. Determinant, rank and inverse of a matrix. Solutions of linear equations. Eigenvalues and eigenvectors of matrices.

**Coordinate geometry** — Straight lines, circles, parabolas, ellipses and hyperbolas. Elements of three dimensional coordinate geometry — straight lines, planes and spheres.

**Calculus** — Sequences and series. Power series. Taylor and Maclaurin series. Limits and continuity of functions of one or more variables. Differentiation and integration of functions of one variable with applications. Definite integrals. Areas using integrals. Definite integrals as limits of Riemann sums. Maxima and minima. Differentiation of functions of several variables. Double integrals and their applications. Ordinary linear differential equations.

## SAMPLE QUESTIONS

*Note: For each question there are four suggested answers of which only one is correct.*

1. Let $b_1, b_2, \dots b_n$ be $n$ positive real numbers satisfying $b_1+b_2+\cdots+b_n = 1$. Then the minimum value of the expression

$$\frac{x_1 + x_2 + \cdots + x_n}{x_1^{b_1}\, x_2^{b_2} \cdots x_n^{b_n}}$$

where $x_1, x_2, \dots, x_n > 0$, is

(A) $\prod_{i=1}^{n} \left(\frac{1}{b_i}\right)^{b_i}$ (B) $n$ (C) $\frac{n}{2}$ (D) $\prod_{i=1}^{n} b_i^{b_i}$.

2. Let $x$ be a positive real number. Then

(A) $x^2 + \pi^2 + x^{2\pi} > x\pi + (\pi + x)x^{\pi}$

(B) $x^{\pi} + \pi^{x} > x^{2\pi} + \pi^{2x}$

(C) $\pi x + (\pi + x)x^{\pi} > x^2 + \pi^2 + x^{2\pi}$

(D) none of the above.

3. A club with $x$ members is organized into four committees such that

(a) each member is in exactly two committees,

(b) any two committees have exactly one member in common.

Then $x$ has

(A) exactly two values both between 4 and 8
(B) exactly one value and this lies between 4 and 8
(C) exactly two values both between 8 and 16
(D) exactly one value and this lies between 8 and 16.

4. The number of ways in which six digits, $1, 2, \ldots, 6$ respectively, can be assigned to six faces of a cube (without repetition of digits) so that one arrangement cannot be obtained from another by a rotation of the cube is

(A) 24 (B) 30 (C) 120 (D) 720.

5. Suppose $X = \{2, 3, 7, 10\}$ and $Y = \{1, 2, 5, 6\}$. The number of pairs $(A, B)$ of nonempty subsets $A \subseteq X$ and $B \subseteq Y$ so that the set $\{a + b : a \in A, b \in B\}$ contains only even integers, is

(A) 9 (B) 18 (C) 32 (D) None of the these.

6. For a pair $(A, B)$ of subsets of the set $X = \{1, 2, \ldots, 100\}$, let $A \triangle B$ denote the set of all elements of $X$ which belong to exactly one of $A$ or $B$. The number of pairs $(A, B)$ of subsets of $X$ such that $A \triangle B = \{2, 4, 6, \ldots, 100\}$ is

(A) $2^{151}$ (B) $2^{102}$ (C) $2^{101}$ (D) $2^{100}$.

7. Let $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, $n$ being a positive integer. The value of

$$\left(1 + \frac{C_0}{C_1}\right)\left(1 + \frac{C_1}{C_2}\right) \cdots \left(1 + \frac{C_{n-1}}{C_n}\right)$$

is

(A) $\left(\frac{n+1}{n+2}\right)^n$ (B) $\frac{n^n}{n!}$ (C) $\left(\frac{n}{n+1}\right)^n$ (D) $\frac{(n+1)^n}{n!}$.

8. $x^2 + x + 1$ is a factor of $(x+1)^n - x^n - 1$, whenever

(A) $n$ is odd
(B) $n$ is odd and a multiple of 3
(C) $n$ is an even multiple of 3
(D) $n$ is odd and not a multiple of 3.

9. The equation $x^6 - 5x^4 + 16x^2 - 72x + 9 = 0$ has

(A) exactly two distinct real roots
(B) exactly three distinct real roots
(C) exactly four distinct real roots
(D) six distinct real roots.

10. The number of real roots of the equation

$$2\cos(\frac{x^2+x}{6}) = 2^x + 2^{-x}$$

is

(A) 0. (B) 1. (C) 2. (D) infinitely many.

11. Consider the following system of equivalences of integers.

$$\begin{aligned} x &\equiv 2 \bmod 15 \\ x &\equiv 4 \bmod 21. \end{aligned}$$

The number of solutions in $x$, where $1 \le x \le 315$, to the above system of equivalences is

(A) 0 (B) 1 (C) 2 (D) 3.

12. The number of real roots of the equation

$$\sqrt[4]{97-x} + \sqrt[4]{x} = 5,$$

is

(A) 4 (B) 3 (C) 2 (D) 1.

13. If two real polynomials $f(x)$ and $g(x)$ of degrees $m\,(\ge 2)$ and $n\,(\ge 1)$ respectively, satisfy

$$f(x^2+1) = f(x)\,g(x),$$

for every $x \in \mathbb{R}$, then

(A) $f$ has exactly one real root $x_0$ such that $f'(x_0) \neq 0$
(B) $f$ has exactly one real root $x_0$ such that $f'(x_0) = 0$
(C) $f$ has $m$ distinct real roots
(D) $f$ has no real root.

14. Let $x_1, x_2, \ldots, x_n$ be $n$ constants each taking values either $-1$ or 1. Next, define $x_{n+1} = x_1$. If $\sum_{i=1}^{n} x_i x_{i+1} = 0$, then

(A) $n$ can be any even number
(B) $n$ must be divisible by 4
(C) $n$ must be divisible by 8
(D) None of the above.

15. Let $X = \dfrac{1}{1001} + \dfrac{1}{1002} + \dfrac{1}{1003} + \cdots + \dfrac{1}{3001}$. Then,

(A) $X < 1$ (B) $X > 3/2$

(C) $1 < X < 3/2$ (D) none of the above holds.

16. The set of complex numbers $z$ satisfying the equation

$$(3+7i)z + (10-2i)\overline{z} + 100 \quad = \quad 0$$

represents, in the complex plane,

(A) a straight line

(B) a pair of intersecting straight lines

(C) a point

(D) a pair of distinct parallel straight lines.

17. The limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} |e^{\frac{2\pi i k}{n}} - e^{\frac{2\pi i (k-1)}{n}}|$$

is

(A) 2 (B) $2e$ (C) $2\pi$ (D) $2i$.

18. Let $\omega$ denote a complex fifth root of unity. Define

$$b_k = \sum_{j=0}^{4} j\,\omega^{-kj},$$

for $0 \le k \le 4$. Then $\sum_{k=0}^{4} b_k\,\omega^k$ is equal to

(A) 5 (B) $5\omega$ (C) $5(1+\omega)$ (D) 0.

19. The value of

$$\lim_{N\to\infty} \frac{\displaystyle\sum_{i=1}^{N} \frac{\log_e i}{i}}{(\log_e N)^2}$$

is

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) None of these.

20. Let $X$ be a nonempty set and let $\mathcal{P}(X)$ denote the collection of all subsets of $X$. Define $f : X \times \mathcal{P}(X) \to \mathbb{R}$ by

$$f(x, A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Then $f(x, A \cup B)$ equals

(A) $f(x, A) + f(x, B)$
(B) $f(x, A) + f(x, B) - 1$
(C) $f(x, A) + f(x, B) - f(x, A) \cdot f(x, B)$
(D) $f(x, A) + |f(x, A) - f(x, B)|$

21. The set $\{x : |x + 1/x| > 6\}$ equals the set
(A) $(0, 3 - 2\sqrt{2}) \cup (3 + 2\sqrt{2}, \infty)$
(B) $(-\infty, -3 - 2\sqrt{2}) \cup (-3 + 2\sqrt{2}, \infty)$
(C) $(-\infty, 3 - 2\sqrt{2}) \cup (3 + 2\sqrt{2}, \infty)$
(D) $(-\infty, -3 - 2\sqrt{2}) \cup (-3 + 2\sqrt{2}, 3 - 2\sqrt{2}) \cup (3 + 2\sqrt{2}, \infty)$

22. Consider the function $f : [0, 1) \longrightarrow [0, 1)$ given by

$$f(x) = (x + 0.5) \mod 1$$

that is, the fractional part of $(x + 0.5)$. Also, for any subset $A \subseteq [0, 1)$, define $f^{-1}(A) = \{x \in [0, 1) : f(x) \in A\}$. If $A = [0,\ \frac{1}{4}) \cup [\frac{1}{2},\ \frac{3}{4})$ then

(A) $f^{-1}\left([\frac{1}{2},\ \frac{3}{4})\right) = A$
(B) $f^{-1}(A) = [0,\ 1/4)$
(C) $f^{-1}(A) = A$
(D) None of the above.

23. Consider the sets defined by the real solutions of the inequalities

$$A = \{(x, y) : x^2 + y^4 \leq 1\}$$
$$B = \{(x, y) : x^4 + y^6 \leq 1\}.$$

Then

(A) $B \subseteq A$
(B) $A \subseteq B$
(C) Each of the sets $A - B$, $B - A$ and $A \cap B$ is non-empty
(D) None of the above.

24. If $f(x)$ is a real valued function such that

$$2f(x) + 3f(-x) = 15 - 4x,$$

for every $x \in \mathbb{R}$, then $f(2)$ is

(A) $-15$ (B) 22 (C) 11 (D) 0.

25. If $f(x) = \dfrac{\sqrt{3}\sin x}{2 + \cos x}$, then the range of $f(x)$ is

(A) the interval $[-1, \sqrt{3}/2]$
(B) the interval $[-\sqrt{3}/2, 1]$
(C) the interval $[-1, 1]$
(D) none of the above.

26. If $f(x) = x^2$ and $g(x) = x \sin x + \cos x$ then

(A) $f$ and $g$ agree at no points

(B) $f$ and $g$ agree at exactly one point

(C) $f$ and $g$ agree at exactly two points

(D) $f$ and $g$ agree at more than two points.

27. For non-negative integers $m, n$ define a function as follows

$$f(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ f(m-1,1) & \text{if } m \neq 0, n = 0 \\ f(m-1, f(m, n-1)) & \text{if } m \neq 0, n \neq 0 \end{cases}$$

Then the value of $f(1,1)$ is

(A) 4 (B) 3 (C) 2 (D) 1.

28. The rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$ is less than 4 if and only if

(A) $a = b = c = d$

(B) at least two of $a, b, c, d$ are equal

(C) at least three of $a, b, c, d$, are equal

(D) $a, b, c, d$ are distinct real numbers.

29. If $M$ is a $3 \times 3$ matrix such that

$$[0\ \ 1\ \ 2]\, M = [1\ \ 0\ \ 0] \quad \text{and} \quad [3\ \ 4\ \ 5]\, M = [0\ \ 1\ \ 0],$$

then $[6\ \ 7\ \ 8]\, M$ is equal to

(A) $[2\ \ 1\ \ -2]$ (B) $[0\ \ 0\ \ 1]$ (C) $[-1\ \ 2\ \ 0]$ (D) $[9\ \ 10\ \ 8]$.

30. Let $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}.$$

If $\lambda_1 + \lambda_2 + \lambda_3 = \sqrt{2} + 1$, then the set of possible values of $t$, $-\pi \leq t < \pi$, is

(A) Empty set (B) $\{\frac{\pi}{4}\}$ (C) $\{-\frac{\pi}{4}, \frac{\pi}{4}\}$ (D) $\{-\frac{\pi}{3}, \frac{\pi}{3}\}$.

31. Let $P_1, P_2$ and $P_3$ denote, respectively, the planes defined by

$$\begin{aligned} a_1x + b_1y + c_1z &= \alpha_1 \\ a_2x + b_2y + c_2z &= \alpha_2 \\ a_3x + b_3y + c_3z &= \alpha_3. \end{aligned}$$

It is given that $P_1, P_2$ and $P_3$ intersect exactly at one point when $\alpha_1 = \alpha_2 = \alpha_3 = 1$. If now $\alpha_1 = 2, \alpha_2 = 3$ and $\alpha_3 = 4$ then the planes

(A) do not have any common point of intersection

(B) intersect at a unique point

(C) intersect along a straight line

(D) intersect along a plane.

32. The values of $\eta$ for which the following system of equations

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ x & + & 2y & + & 4z & = & \eta \\ x & + & 4y & + & 10z & = & \eta^2 \end{array}$$

has a solution are

(A) $\eta = 1, -2$ (B) $\eta = -1, -2$ (C) $\eta = 3, -3$ (D) $\eta = 1, 2$.

33. In a rectangle $ABCD$, the co-ordinates of $A$ and $B$ are (1, 2) and (3, 6) respectively and some diameter of the circle circumscribing $ABCD$ has the equation $2x - y + 4 = 0$. Then the area of the rectangle $ABCD$ is

(A) 16 (B) $2\sqrt{10}$ (C) $2\sqrt{5}$ (D) 20.

34. If the tangent at the point $P$ with co-ordinates $(h, k)$ on the curve $y^2 = 2x^3$ is perpendicular to the straight line $4x = 3y$, then

(A) $(h, k) = (0, 0)$

(B) $(h, k) = (1/8, -1/16)$

(C) $(h, k) = (0, 0)$ or $(h, k) = (1/8, -1/16)$

(D) no such point $(h, k)$ exists.

35. Consider a family of straight lines

$$ax + by - 49 = 0,$$

where $a^2 + b^2 = 1$. Then the curve which touches each of these straight lines at a single point is

(A) a circle with center (0,0) and radius 7

(B) an ellipse with center at (0,0) with the semi-axes 7 and 49

(C) a circle with center (0,0) and radius 49

(D) $(x \pm 49)^2 + (y \pm 49)^2 = 49$.

36. Suppose the circle with equation $x^2 + y^2 + 2fx + 2gy + c = 0$ cuts the parabola $y^2 = 4ax,\ (a > 0)$ at four distinct points. If $d$ denotes the sum of ordinates of these four points, then the set of possible values of $d$ is

(A) $\{0\}$ (B) $(-4a,\, 4a)$ (C) $(-a,\, a)$ (D) $(-\infty, \infty)$.

37. If a sphere of radius $r$ passes through the origin and cuts the three co-ordinate axes at points $A, B, C$ respectively, then the centroid of the triangle $ABC$ lies on a sphere of radius

(A) $r$ (B) $\frac{r}{\sqrt{3}}$ (C) $\sqrt{\frac{2}{3}}\, r$ (D) $\frac{2r}{3}$.

38. Consider the tangent plane $\mathcal{T}$ at the point $(1/\sqrt{3},\, 1/\sqrt{3},\, 1/\sqrt{3})$ to the sphere $x^2 + y^2 + z^2 = 1$. If $P$ is an arbitrary point on the plane

$$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = -2,$$

then the minimum distance of $P$ from the tangent plane, $\mathcal{T}$, is always

(A) $\sqrt{5}$ (B) 3 (C) 1 (D) None of these.

39. Let $S_1$ denote a sphere of unit radius and $C_1$ a cube inscribed in $S_1$. Inductively define spheres $S_n$ and cubes $C_n$ such that $S_{n+1}$ is inscribed in $C_n$ and $C_{n+1}$ is inscribed in $S_{n+1}$. Let $v_n$ denote the sum of the volumes of the first $n$ spheres. Then $\lim_{n\to\infty} v_n$ is

(A) $2\pi$. (B) $\frac{8\pi}{3}$. (C) $\frac{2\pi}{13}(9 + \sqrt{3})$. (D) $\frac{6+2\sqrt{3}}{3}\pi$.

40. If $0 < x < 1$, then the sum of the infinite series

$$\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \cdots$$

is

(A) $\log \frac{1+x}{1-x}$

(B) $\frac{x}{1-x} + \log(1+x)$

(C) $\frac{1}{1-x} + \log(1-x)$

(D) $\frac{x}{1-x} + \log(1-x)$.

41. Let $\{a_n\}$ be a sequence of real numbers. Then $\lim\limits_{n\to\infty} a_n$ exists if and only if

(A) $\lim\limits_{n\to\infty} a_{2n}$ and $\lim\limits_{n\to\infty} a_{2n+2}$ exists
(B) $\lim\limits_{n\to\infty} a_{2n}$ and $\lim\limits_{n\to\infty} a_{2n+1}$ exist
(C) $\lim\limits_{n\to\infty} a_{2n}$, $\lim\limits_{n\to\infty} a_{2n+1}$ and $\lim\limits_{n\to\infty} a_{3n}$ exist
(D) none of the above.

42. Let $\{a_n\}$ be a sequence of non-negative real numbers such that the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent. If $p$ is a real number such that the series $\sum \frac{\sqrt{a_n}}{n^p}$ diverges, then

(A) $p$ must be strictly less than $\frac{1}{2}$
(B) $p$ must be strictly less than or equal to $\frac{1}{2}$
(C) $p$ must be strictly less than or equal to 1 but can be greater than $\frac{1}{2}$
(D) $p$ must be strictly less than 1 but can be greater than or equal to $\frac{1}{2}$.

43. In the Taylor expansion of the function $f(x) = e^{x/2}$ about $x = 3$, the coefficient of $(x-3)^5$ is

(A) $e^{3/2}\frac{1}{5!}$ (B) $e^{3/2}\frac{1}{2^5 5!}$ (C) $e^{-3/2}\frac{1}{2^5 5!}$ (D) none of the above.

44. Suppose $a > 0$. Consider the sequence

$$a_n = n\{\sqrt[n]{ea} - \sqrt[n]{a}\}, \quad n \geq 1.$$

Then

(A) $\lim_{n\to\infty} a_n$ does not exist
(B) $\lim_{n\to\infty} a_n = e$
(C) $\lim_{n\to\infty} a_n = 0$
(D) none of the above.

45. Let $\{a_n\}$, $n \geq 1$, be a sequence of real numbers satisfying $|a_n| \leq 1$ for all $n$. Define

$$A_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n),$$

for $n \geq 1$. Then $\lim\limits_{n\to\infty} \sqrt{n}(A_{n+1} - A_n)$ is equal to

(A) 0 (B) $-1$ (C) 1 (D) None of these.

46. Let $x_n = \dfrac{n+1}{n+5}$ for $n = 1, 2, 3, \ldots$. For each $\epsilon > 0$, define

$$N(\epsilon) = \min\{k : |x_n - 1| < \epsilon \text{ for all } n \geq k\}.$$

Then $N(\frac{1}{1000})$ is

(A) greater than 3000
(B) less than 1000
(C) equal to 2500
(D) None of the above.

47. The smallest positive number $K$ for which the inequality

$$|\sin^2 x - \sin^2 y| \leq K|x-y|$$

holds for all $x$ and $y$ is

(A) 2 (B) 1 (C) $\frac{\pi}{2}$
(D) there is no smallest positive value of $K$; any $K > 0$ will make the inequality hold.

48. Given two real numbers $a < b$, let

$$d(x,[a,b]) = \min\{|x-y| \ : a \leq y \leq b\} \text{ for } -\infty < x < \infty.$$

Then the function

$$f(x) = \frac{d(x,[0,\ 1])}{d(x,[0,\ 1]) + d(x,[2,\ 3])}$$

satisfies

(A) $0 \leq f(x) < \frac{1}{2}$ for every $x$
(B) $0 < f(x) < 1$ for every $x$
(C) $f(x) = 0$ if $2 \leq x \leq 3$ and $f(x) = 1$ if $0 \leq x \leq 1$
(D) $f(x) = 0$ if $0 \leq x \leq 1$ and $f(x) = 1$ if $2 \leq x \leq 3$.

49. Let

$$f(x,y) = \begin{cases} e^{-1/(x^2+y^2)} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then $f(x,y)$ is

(A) not continuous at $(0,0)$
(B) continuous at $(0,0)$ but does not have first order partial derivatives
(C) continuous at $(0,0)$ and has first order partial derivatives, but not differentiable at $(0,0)$
(D) differentiable at $(0,0)$

50. Let $f(x)$ be the function

$$f(x) = \begin{cases} \dfrac{x^p}{(\sin x)^q} & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then $f(x)$ is continuous at $x = 0$ if

(A) $p > q$ (B) $p > 0$ (C) $q > 0$ (D) $p < q$.

51. Let $w = \log(u^2 + v^2)$ where $u = \mathrm{e}^{(x^2+y)}$ and $v = \mathrm{e}^{(x+y^2)}$. Then

$$\left.\frac{\partial w}{\partial x}\right|_{x=0,\ y=0}$$

is

(A) 0 (B) 1 (C) 2 (D) 4

52. Let $p > 1$ and for $x > 0$, define $f(x) = (x^p - 1) - p(x - 1)$. Then

(A) $f(x)$ is an increasing function of $x$ on $(0, \infty)$

(B) $f(x)$ is a decreasing function of $x$ on $(0, \infty)$

(C) $f(x) \geq 0$ for all $x > 0$

(D) $f(x)$ takes both positive and negative values for $x \in (0, \infty)$.

53. The map $f(x) = a_0 \cos|x| + a_1 \sin|x| + a_2|x|^3$ is differentiable at $x = 0$ if and only if

(A) $a_1 = 0$ and $a_2 = 0$ (B) $a_0 = 0$ and $a_1 = 0$

(C) $a_1 = 0$ (D) $a_0, a_1, a_2$ can take any real value.

54. $f(x)$ is a differentiable function on the real line such that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} f'(x) = \alpha$. Then

(A) $\alpha$ must be 0 (B) $\alpha$ need not be 0, but $|\alpha| < 1$

(C) $\alpha > 1$ (D) $\alpha < -1$.

55. Let $f$ and $g$ be two differentiable functions such that $f'(x) \leq g'(x)$ for all $x < 1$ and $f'(x) \geq g'(x)$ for all $x > 1$. Then

(A) if $f(1) \geq g(1)$, then $f(x) \geq g(x)$ for all $x$

(B) if $f(1) \leq g(1)$, then $f(x) \leq g(x)$ for all $x$

(C) $f(1) \leq g(1)$

(D) $f(1) \geq g(1)$.

56. The length of the curve $x = t^3$, $y = 3t^2$ from $t = 0$ to $t = 4$ is

(A) $5\sqrt{5} + 1$ (B) $8(5\sqrt{5} + 1)$

(C) $5\sqrt{5} - 1$ (D) $8(5\sqrt{5} - 1)$.

57. Let

$$f(x) = \begin{cases} x & \text{if} \quad x \in [0,2] \\ 0 & \text{if} \quad x \notin [0,2] \end{cases}$$

$$g(y) = \begin{cases} 1 & \text{if} \quad x \in [0,2] \\ 0 & \text{if} \quad x \notin [0,2]. \end{cases}$$

Let $A = \{(x, y) : x + y \leq 3\}$. Then the value of the integral

$$\int\int_A f(x)g(y)\ dx\ dy$$

equals

(A) $\frac{9}{2}$ (B) $\frac{7}{2}$ (C) 4 (D) $\frac{19}{6}$.

58. Let $0 < \alpha < \beta < 1$. Then

$$\sum_{k=1}^{\infty} \int_{\frac{1}{k+\beta}}^{\frac{1}{k+\alpha}} \frac{1}{1+x} dx$$

is equal to

(A) $\log_e \dfrac{\beta}{\alpha}$ (B) $\log_e \dfrac{1+\beta}{1+\alpha}$ (C) $\log_e \dfrac{1+\alpha}{1+\beta}$ (D) $\infty$.

59. If $f$ is continuous in [0,1] then

$$\lim_{n\to\infty} \sum_{j=0}^{[n/2]} \frac{1}{n} f\left(\frac{j}{n}\right)$$

(where $[y]$ is the largest integer less than or equal to $y$)

(A) does not exist

(B) exists and is equal to $\frac{1}{2}\int_0^1 f(x)\, dx$

(C) exists and is equal to $\int_0^1 f(x)\, dx$

(D) exists and is equal to $\int_0^{1/2} f(x)\, dx$.

60. The volume of the solid, generated by revolving about the horizontal line $y = 2$ the region bounded by $y^2 \leq 2x$, $x \leq 8$ and $y \geq 2$, is

(A) $2\sqrt{2\pi}$ (B) $28\pi/3$ (C) $84\pi$ (D) none of the above.

61. The minimum value of

$$(\sqrt{3}\cos\theta + \sin\theta)(\sin\theta + \cos\theta)$$

in the interval $(0, \pi/2)$ is attained

(A) at exactly one point (B) at exactly two points

(C) at exactly three points (D) nowhere.

62. Given a set of $n$ variables $x_1, x_2, \ldots, x_n$, where $x_i \in [0,1]$ for $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} x_i = 1$. Let the function

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i(1 - x_i)$$

attain its maximum value when $x_1 = x_1^0,\ x_2 = x_2^0, \ldots, x_n = x_n^0$. Then

(A) $x_1^0 = x_2^0 = \ldots = x_n^0 = \frac{1}{n}$

(B) no two values among $x_1^0, x_2^0, \ldots, x_n^0$ are equal

(C) only one $x_i^0 = 1$ and the others are zeros

(D) none of the above.

63. The coordinates of a moving point $P$ satisfy the equations

$$\frac{dx}{dt} = \tan x, \quad \frac{dy}{dt} = -\sin^2 x, \quad t \geq 0.$$

If the curve passes through the point $(\pi/2, 0)$ when $t = 0$, then the equation of the curve in rectangular co-ordinates is

(A) $y = 1/2\cos^2 x$ (B) $y = \sin 2x$

(C) $y = \cos 2x + 1$ (D) $y = \sin^2 x - 1$.

64. Let $y$ be a function of $x$ satisfying

$$\frac{dy}{dx} = 2x^3\sqrt{y} - 4xy$$

If $y(0) = 0$ then $y(1)$ equals

(A) $1/4e^2$ (B) $1/e$ (C) $e^{1/2}$ (D) $e^{3/2}$.

65. Let $f(x)$ be a given differentiable function. Consider the following differential equation in $y$

$$f(x)\frac{dy}{dx} = yf'(x) - y^2. \qquad (1)$$

The general solution of this equation is given by

(A) $y = -\dfrac{x+c}{f(x)}$ (B) $y^2 = \dfrac{f(x)}{x+c}$

(C) $y = \dfrac{f(x)}{x+c}$ (D) $y = \dfrac{[f(x)]^2}{x+c}$.

66. The differential equation of the system of circles touching the $y$-axis at the origin is

(A) $x^2 + y^2 - 2xy\,\dfrac{dy}{dx} = 0$

(B) $x^2 + y^2 + 2xy\,\dfrac{dy}{dx} = 0$

(C) $x^2 - y^2 - 2xy\,\dfrac{dy}{dx} = 0$

(D) $x^2 - y^2 + 2xy\,\dfrac{dy}{dx} = 0.$

67. Let $y(x)$ be a non-trivial solution of the second order linear differential equation

$$\frac{d^2y}{dx^2} + 2c\frac{dy}{dx} + ky = 0,$$

where $c < 0$, $k > 0$ and $c^2 > k$. Then

(A) $|y(x)| \to \infty$ as $x \to \infty$

(B) $|y(x)| \to 0$ as $x \to \infty$

(C) $\lim_{x\to\pm\infty} |y(x)|$ exists and is finite

(D) none of the above is true.

68. Suppose a solution of the differential equation

$$(x\,y^3 + x^2\,y^7)\,\frac{dy}{dx} = 1,$$

satisfies the initial condition $y(\frac{1}{4}) = 1$. Then the value of $\dfrac{dy}{dx}$ when $y = -1$ is

(A) $\frac{4}{3}$ (B) $-\frac{4}{3}$ (C) $\frac{16}{5}$ (D) $-\frac{16}{5}$.

69. Consider the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbf{R},\ a > 0 \right\}$$

with usual matrix multiplication. Let

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbf{R} \right\}.$$

Then,

(A) $N$ is not a subgroup of $G$

(B) $N$ is a subgroup of $G$ but not a normal subgroup

(C) $N$ is a normal subgroup and the quotient group $G/N$ is of finite order

(D) $N$ is a normal subgroup and the quotient group is isomorphic to $\mathbf{R}^+$ (the group of positive reals with multiplication).

70. Let $S_n$ be the group of permutations on $n$ symbols. Then

(A) $S_4$ has no subgroup isomorphic to $S_3$

(B) $S_4$ has only one subgroup isomorphic to $S_3$

(C) $S_4$ has exactly 3 distinct subgroups isomorphic to $S_3$

(D) $S_4$ has 4 subgroups isomorphic to $S_3$.