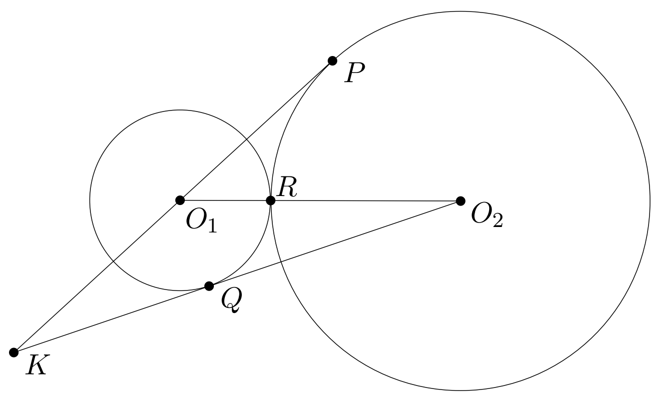
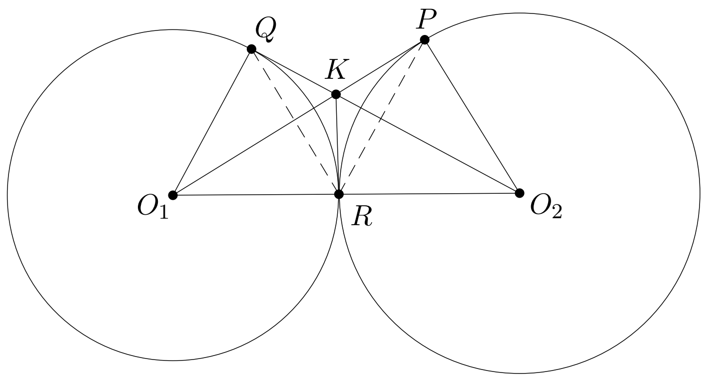
Problems and solutions: INMO 2013

Problem 1. *Let Γ_1 and Γ_2 be two circles touching each other externally at R. Let l_1 be a line which is tangent to Γ_2 at P and passing through the center O_1 of Γ_1. Similarly, let l_2 be a line which is tangent to Γ_2 at Q and passing through the center O_2 of Γ_2. Suppose l_1 and l_2 are not parallel and interesct at K. If $KP = KQ$, prove that the triangle PQR is equilateral.*

Solution. Suppose that P and Q lie on the opposite sides of line joining O_1 and O_2. By symmetry we may assume that the configuration is as shown in the figure below. Then we have $KP > KO_1 > KQ$ since KO_1 is the hypotenuse of triangle KQO_1. This is a contradiction to the given assumption, and therefore P and Q lie on the same side of the line joining O_1 and O_2.



Since $KP = KQ$ it follows that K lies on the radical axis of the given circles, which is the common tangent at R. Therefore $KP = KQ = KR$ and hence K is the cirumcenter of $\triangle PQR$.



On the other hand, $\triangle KQO_1$ and $\triangle KRO_1$ are both right-angled triangles with $KQ = KR$ and $QO_1 = RO_1$, and hence the two triangles are congruent. Therefore $\widehat{QKO_1} = \widehat{RKO_1}$, so KO_1, and hence PK is perpendicular to QR. Similarly, QK is perpendicular to PR, so it follows that K is the orthocenter of $\triangle PQR$. Hence we have that $\triangle PQR$ is equilateral.

□

Alternate solution. We again rule out the possibility that P and Q are on the opposite side of the line joining O_1O_2, and assume that they are on the same side.

Observe that $\triangle KPO_2$ is congruent to $\triangle KQO_1$ (since $KP = KQ$). Therefore $O_1P = O_2Q = r$ (say). In $\triangle O_1O_2Q$, we have $\widehat{O_1QO_2} = \pi/2$ and R is the midpoint of the hypotenuse, so $RQ = RO_1 = r$. Therefore $\triangle O_1RQ$ is equilateral, so $\widehat{QRO_1} = \pi/3$. Similarly, $PR = r$ and $\widehat{PRO_2} = \pi/3$, hence $\widehat{PRQ} = \pi/3$. Since $PR = QR$ it follows that $\triangle PQR$ is equilateral. □

Problem 2. *Find all positive integers m, n, and primes $p \geq 5$ such that*

$$m(4m^2 + m + 12) = 3(p^n - 1)\,.$$

Solution. Rewriting the given equation we have

$$4m^3 + m^2 + 12m + 3 = 3p^n\,.$$

The left hand side equals $(4m+1)(m^2+3)$.

Suppose that $(4m+1, m^2+3) = 1$. Then $(4m+1, m^2+3) = (3p^n, 1), (3, p^n), (p^n, 3)$ or $(1, 3p^n)$, a contradiction since $4m+1, m^2+3 \geq 4$. Therefore $(4m+1, m^2+3) > 1$.

Since $4m+1$ is odd we have $(4m+1, m^2+3) = (4m+1, 16m^2+48) = (4m+1, 49) = 7$ or 49. This proves that $p = 7$, and $4m+1 = 3\cdot 7^k$ or 7^k for some natural number k. If $(4m+1, 49) = 7$ then we have $k = 1$ and $4m+1 = 21$ which does not lead to a solution. Therefore $(4m+1, m^2+3) = 49$. If 7^3 divides $4m+1$ then it does not divide m^2+3, so we get $m^2+3 \leq 3\cdot 7^2 < 7^3 \leq 4m+1$. This implies $(m-2)^2 < 2$, so $m \leq 3$, which does not lead to a solution. Therefore we have $4m+1 = 49$ which implies $m = 12$ and $n = 4$. Thus $(m, n, p) = (12, 4, 7)$ is the only solution. □

Problem 3. *Let a, b, c, d be positive integers such that $a \geq b \geq c \geq d$. Prove that the equation $x^4 - ax^3 - bx^2 - cx - d = 0$ has no integer solution.*

Solution. Suppose that m is an integer root of $x^4 - ax^3 - bx^2 - cx - d = 0$. As $d \neq 0$, we have $m \neq 0$. Suppose now that $m > 0$. Then $m^4 - am^3 = bm^2 + cm + d > 0$ and hence $m > a \geq d$. On the other hand $d = m(m^3 - am^2 - bm - c)$ and hence m divides d, so $m \leq d$, a contradiction. If $m < 0$, then writing $n = -m > 0$ we have $n^4 + an^3 - bn^2 + cn - d = n^4 + n^2(an - b) + (cn - d) > 0$, a contradiction. This proves that the given polynomial has no integer roots. □

Problem 4. *Let n be a positive integer. Call a nonempty subset S of $\{1, 2, \ldots, n\}$* good *if the arithmetic mean of the elements of S is also an integer. Further let t_n denote the number of good subsets of $\{1, 2, \ldots, n\}$. Prove that t_n and n are both odd or both even.*

Solution. We show that $T_n - n$ is even. Note that the subsets $\{1\}, \{2\}, \cdots, \{n\}$ are good. Among the other good subsets, let A be the collection of subsets with an integer average which belongs to the subset, and let B be the collection of subsets with an integer average which is not a member of the subset. Then there is a bijection between A and B, because removing the average takes a member of A to a member of B; and including the average in a member of B takes it to its inverse. So $T_n - n = |A| + |B|$ is even. □

Alternate solution. Let $S = \{1, 2, \ldots, n\}$. For a subset A of S, let $\overline{A} = \{n + 1 - a | a \in A\}$. We call a subset A symmetric if $\overline{A} = A$. Note that the arithmetic mean of a symmetric subset is $(n+1)/2$. Therefore, if n is even, then there are no symmetric good subsets, while if n is odd then every symmetric subset is good.

If A is a proper good subset of S, then so is $\overline{A}$. Therefore, all the good subsets that are not symmetric can be paired. If n is even then this proves that t_n is even. If n is odd, we have to show that there are odd number of symmetric subsets. For this, we note that a symmetric subset contains the element $(n + 1)/2$ if and only if it has odd number of elements. Therefore, for any natural number k, the number of symmetric subsets of size $2k$ equals the number of symmetric subsets of size $2k + 1$. The result now follows since there is exactly one symmetric subset with only one element. □

Problem 5. *In an acute triangle ABC, O is the circumcenter, H is the orthocenter and G is the centroid. Let OD be perpendicular to BC and HE be perpendicular to CA, with D on BC and E on CA. Let F be the midpoint of AB. Suppose the areas of triangles ODC, HEA and GFB are equal. Find all the possible values of $\widehat{C}$.*

Solution. Let R be the circumradius of $\triangle ABC$ and Δ its area. We have $OD = R\cos A$ and $DC = \frac{a}{2}$, so

$$[ODC] = \frac{1}{2} \cdot OD \cdot DC = \frac{1}{2} \cdot R\cos A \cdot R\sin A = \frac{1}{2}R^2 \sin A \cos A\,. \tag{1}$$

Again $HE = 2R\cos C \cos A$ and $EA = c\cos A$. Hence

$$[HEA] = \frac{1}{2} \cdot HE \cdot EA = \frac{1}{2} \cdot 2R\cos C \cos A \cdot c\cos A = 2R^2 \sin C \cos C \cos^2 A\,. \tag{2}$$

Further

$$[GFB] = \frac{\Delta}{6} = \frac{1}{6} \cdot 2R^2 \sin A \sin B \sin C = \frac{1}{3}R^2 \sin A \sin B \sin C\,. \tag{3}$$

Equating (1) and (2) we get $\tan A = 4\sin C\cos C$. And equating (1) and (3), and using this relation we get

$$\begin{aligned} 3\cos A &= 2\sin B\sin C = 2\sin(C+A)\sin C \\ &= 2(\sin C + \cos C\tan A)\sin C\cos A \\ &= 2\sin^2 C(1+4\cos^2 C)\cos A\,. \end{aligned}$$

Since $\cos A \neq 0$ we get $3 = 2t(-4t+5)$ where $t = \sin^2 C$. This implies $(4t-3)(2t-1) = 0$ and therefore, since $\sin C > 0$, we get $\sin C = \sqrt{3}/2$ or $\sin C = 1/\sqrt{2}$. Because $\triangle ABC$ is acute, it follows that $\widehat{C} = \pi/3$ or $\pi/4$.

We observe that the given conditions are satisfied in an equilateral triangle, so $\widehat{C} = \pi/3$ is a possibility. Also, the conditions are satisfied in a triangle where $\widehat{C} = \pi/4, \widehat{A} = \tan^{-1} 2$ and $\widehat{B} = \tan^{-1} 3$. Therefore $\widehat{C} = \pi/4$ is also a possibility.

Thus the two possible values of $\widehat{C}$ are $\pi/3$ and $\pi/4$. $\square$

Problem 6. *Let a, b, c, x, y, z be positive real numbers such that $a+b+c = x+y+z$ and $abc = xyz$. Further, suppose that $a \le x < y < z \le c$ and $a < b < c$. Prove that $a = x, b = y$ and $c = z$.*

Solution. Let

$$f(t) = (t-x)(t-y)(t-z) - (t-a)(t-b)(t-c)\,.$$

Then $f(t) = kt$ for some constant k. Note that $ka = f(a) = (a-x)(a-y)(a-z) \le 0$ and hence $k \le 0$. Similarly, $kc = f(c) = (c-x)(c-y)(c-z) \ge 0$ and hence $k \ge 0$. Combining the two, it follows that $k = 0$ and that $f(a) = f(c) = 0$. These equalities imply that $a = x$ and $c = z$, and then it also follows that $b = y$. $\square$