SAMPLE QUESTION PAPER
Mathematics - Class XII (Code A)

Time : 3 Hours **Max. Marks : 100**

General Instructions :

(i) All questions are compulsory.

(ii) The question paper consist of 29 questions divided into three sections A, B and C. **Section A** comprises of 10 questions of **one mark** each, **section B** comprises of 12 questions of **four marks** each and **section C** comprises of 7 questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, Internal choice has been provided in 4 questions of four marks and 2 questions of six marks. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculator is not permitted. You may ask for logarithmic tables, if required.

SECTION - A

1. Show that the function $f : R \to R$, defined by $f(x) = x^3$ is one one. **[1]**

2. Show that the operation '$*$' on R defined as $a*b = \max(a, b)$ is a binary operation on R. **[1]**

3. If $\sin^{-1}x + \sin^{-1}\left(\frac{5}{13}\right) = \frac{\pi}{2}$, then find value of '$x$'. **[1]**

4. Find A^2, where $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$. **[1]**

5. Find the value of k for which the following function is continuous at $x = 1$. **[1]**

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ k, & x = 1 \end{cases}$$

6. Let $\phi(x) = \lambda x^2 + 7x - 4$, if $\phi'(5) = 97$, find λ. **[1]**

7. An edge of a variable cube is increasing at the rate of 10 cm/s. How fast the volume of the cube is increasing when the edge is 5 cm long? **[1]**

8. Find the slope of tangent to the curve $y = a^{2x-1}$, $a > 0$ at $x = \frac{1}{2}$. **[1]**

9. Evaluate $\int_0^3 [x]dx$, where $[\cdot]$ represents greatest integral function. **[1]**

10. Find the degree of the differential equation $\dfrac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = k.$ **[1]**

SECTION - B

11. Find the value of α for which the function $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is the inverse of itself. **[4]**

12. Solve the equation **[4]**

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

13. Show that the function f defined as $f(x) = 2x - |x|$ is continuous at $x = 0$. **[4]**

14. If $y = \left(x + \sqrt{x^2 + a^2}\right)^n$, then prove that $\dfrac{dy}{dx} = \dfrac{ny}{\sqrt{x^2 + a^2}}$. **[4]**

OR

If $x^2 + y^2 = t - \dfrac{1}{t}$ and $x^4 + y^4 = t^2 + \dfrac{1}{t^2}$, then prove that $\dfrac{dy}{dx} = \dfrac{1}{x^3 y}$.

15. Find the equation of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1). **[4]**

OR

Find the intervals in which the function f defined by $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ is strictly increasing or decreasing.

16. Evaluate $\displaystyle\int \frac{1}{1 - 2\sin x}\,dx$. **[4]**

OR

Evaluate $\displaystyle\int \frac{1}{\sqrt{3}\sin x + \cos x}\,dx$.

17. If a, b, c are different and $\Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, then show that $1 + abc = 0$. **[4]**

18. Evaluate $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}\,dx$. **[4]**

OR

Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$.

19. Prove that the relation R on the set $N \times N$ defined by $(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$ for all (a, b), $(c, d) \in N \times N$ is an equivalence relation. **[4]**

20. If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, $\vec{a} \neq \vec{0}$ then show that $\vec{b} = \vec{c}$. **[4]**

21. Find the shortest distance between the lines $\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$. **[4]**

22. A bag contains 5 white, 7 red and 8 black balls. Four balls are drawn one by one with replacement. Find the probability that at least one ball is white. **[4]**

SECTION - C

23. Find the inverse of the following matrix using elementary operations . **[6]**

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

24. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart. **[6]**

OR

In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product. If the bolt drawn is found to be defective, what is the probability that it is manufactured by the machine B?

25. A house wife wishes to mix together two kinds of food, X and Y, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg of food is given below. **[6]**

	Vitamin A	Vitamin B	Vitamin C
Food X :	1	2	3
Food Y :	2	2	1

One kg of food X costs Rs. 6 and one kg of food Y costs Rs.10. Find the least cost of the mixture which will produce the diet.

26. Find the length and the foot of the perpendicular from the point (7, 14, 5) to the plane $2x + 4y - z = 2$. **[6]**

OR

Find the image of the point having position vector $\hat{i} + 3\hat{j} + 4\hat{k}$ in the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$.

27. Sketch the curves and identify the region bounded by the curves $x = \frac{1}{2}$, $x = 2$, $y = \log x$ and $y = 2^x$. Find the area of this region. **[6]**

28. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side $\sqrt{2}a$. **[6]**

29. Solve $\frac{dy}{dx} - 2y = \cos 3x$. **[6]**

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Mathematics - Class XII

SOLUTIONS

SECTION - A

1. Let $x_1, x_2 \in N$ such that $f(x_1) = f(x_2)$, then

$f(x_1) = f(x_2)$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$, (as $x_1 = x_2$ is unique solution)

$\Rightarrow$ f is one-one.

2. We have

$$a * b = \text{maximum of } a \text{ and } b = \begin{cases} a, & \text{if } a > b \\ b, & \text{if } a \le b \end{cases}$$

Thus $a*b \in R$ for all $a, b \in R$, hence '$*$' is binary operation on R

3. $$\sin^{-1} x + \sin^{-1}\left(\frac{5}{13}\right) = \frac{\pi}{2}$$

$$\sin^{-1} x + \cos^{-1}\left(\sqrt{1-\left(\frac{5}{13}\right)^2}\right) = \frac{\pi}{2}$$

$$\sin^{-1} x + \cos^{-1}\left(\frac{12}{13}\right) = \frac{\pi}{2}$$

$$\Rightarrow x = \frac{12}{13}\left(\text{as } \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right)$$

4. We have

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+3\times2 & 1\times3+3\times1 \\ 2\times1+1\times2 & 2\times3+1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 4 & 7 \end{bmatrix}$$

5. Given,

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ k, & x = 1 \end{cases}$$

for $f(x)$ to be continuous

$$\lim_{x \to 1} f(x) = f(1)$$

$$\Rightarrow \lim_{x \to 1} \frac{(x^2 - 1)}{(x - 1)} = k$$

$$\Rightarrow \lim_{x \to 1} \frac{(x + 1)(x - 1)}{(x - 1)} = k$$

$\Rightarrow k = 2$

6. $\phi(x) = \lambda x^2 + 7x - 4$

$\phi'(x) = 2\lambda x + 7$

$\phi'(5) = 2\lambda \times 5 + 7 = 97$

$\Rightarrow \lambda = 9$

7. Let 'a' be the length of an edge of the cube and v be its volume.

Given, $\frac{da}{dt}$ = 10 cm/s

$v = a^3$

$$\frac{dv}{dt} = 3a^2 . \frac{da}{dt}$$

$= 3 \times (5)^2 \times 10$ cm^3/s

$= 750$ cm^3/s

8. $y = a^{2x - 1}$

$$\frac{dy}{dx} = 2 \log a . a^{2x-1}$$

$$\left(\text{Slope at } x = \frac{1}{2}\right) = \left(\frac{dy}{dx}\right)_{x=\frac{1}{2}}$$

$= 2 \log a . (a)^{2.1/2 - 1}$

$= 2 \log a$

9. $\int_0^3 [x]dx$

$$= \int_0^1 0.dx + \int_1^2 1.dx + \int_2^3 2.dx \qquad \begin{Bmatrix} \text{for } 0 < x < 1 & , & [x] = 0 \\ 1 \le x < 2 & , & [x] = 1 \\ 2 \le x < 3 & , & [x] = 2 \end{Bmatrix}$$

$= (x)_1^2 + (2x)_2^3$

$= 2 - 1 + 2(3 - 2)$

$= 1 + 2 = 3$

10. Given,

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = k$$

$$\Rightarrow \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2 \qquad \text{[on squaring]}$$

Clearly, power of $\left(\frac{d^2y}{dx^2}\right)$ is 2, hence degree is 2.

SECTION - B

11. Clearly, $f(x)$ is a bijection from R to R

Now, $fof^{-1}(x) = x$

$f(f^{-1}(x)) = x$

$\Rightarrow 1+\alpha f^{-1}(x) = x$

$\Rightarrow f^{-1}(x) = \frac{x-1}{\alpha}$

It is given that

$f(x) = f^{-1}(x)\ \forall\ x \in R$

$1 + \alpha x = \frac{x-1}{\alpha}\ \forall\ x \in R$

$\Rightarrow \alpha x + 1 = \frac{1}{\alpha}x + \frac{-1}{\alpha}\ \forall\ x \in R$

Comparing the coefficients of like terms, we get

$\Rightarrow \alpha = \frac{1}{\alpha}$ and $1 = \frac{-1}{\alpha}$

$\Rightarrow \alpha = -1$

12. We have

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{x-1}{x-2}\right)=\tan^{-1}1-\tan^{-1}\frac{x+1}{x+2}$$

$$\Rightarrow \tan^{-1}\left(\frac{x-1}{x-2}\right)=\tan^{-1}\left(\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right)$$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{x-1}{x-2}\right) = \tan^{-1}\left(\dfrac{1}{2x+3}\right)$

$\Rightarrow \quad \dfrac{x-1}{x-2} = \dfrac{1}{2x+3}$

$\Rightarrow \quad x = \pm\dfrac{1}{\sqrt{2}}$

13. We have,

$f(x) = 2x - |x| = \begin{cases} 2x - x, & \text{if } x \geq 0 \\ 2x - (-x), & \text{if } x < 0 \end{cases}$

$\Rightarrow \quad f(x) = \begin{cases} x, & x \geq 0 \\ 3x, & x < 0 \end{cases}$

LHL at $x = 0$

$= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3x = \lim_{h \to 0} 3(0 - h) = 0$

RHL at $x = 0$

$= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = \lim_{h \to 0} (0 + h) = 0$

and $f(0) = 0$

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\therefore \quad f(x)$ is continuous at $x = 0$

14. $y = \left(x + \sqrt{x^2 + a^2}\right)^n$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + \sqrt{x^2 + a^2}\right)^n$$

$$\frac{dy}{dx} = n\left(x + \sqrt{x^2 + a^2}\right)^{n-1} . \frac{d}{dx}\left(x + \sqrt{x^2 + a^2}\right)$$

$$\frac{dy}{dx} = n\left(x + \sqrt{x^2 + a^2}\right)^{n-1} \left\{1 + \frac{1 \times 2x}{2\sqrt{x^2 + a^2}}\right\}$$

$$\frac{dy}{dx} = n\left(x + \sqrt{x^2 + a^2}\right)^{n-1} \left\{1 + \frac{x}{\sqrt{x^2 + a^2}}\right\}$$

$$\frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \left\{\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right\}$$

$$= \frac{ny}{\sqrt{x^2 + a^2}}$$

OR

We have,

$$x^2 + y^2 = t - \frac{1}{t}$$

$$\Rightarrow \quad (x^2 + y^2)^2 = \left(t - \frac{1}{t}\right)^2$$

$$\Rightarrow \quad x^4 + y^4 + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$$

$$\Rightarrow \quad x^4 + y^4 + 2x^2y^2 = x^4 + y^4 - 2$$

$$\Rightarrow \quad 2x^2y^2 = -2 \quad \Rightarrow x^2y^2 = -1$$

$$\Rightarrow \quad y^2 = \frac{-1}{x^2}$$

Now, Differentiating w.r.t. x

$$\Rightarrow \quad 2y\frac{dy}{dx} = -(-2)x^{-3}$$

$$\Rightarrow \quad y\frac{dy}{dx} = \frac{1}{x^3} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{x^3y}$$

15. Given

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$$

On differentiating w.r.t. x

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}.\frac{dy}{dx} = 0$$

Now, $\frac{dy}{dx} = -\left(\frac{x^{\frac{-1}{3}}}{y^{\frac{-1}{3}}}\right)$

$$\left(\frac{dy}{dx}\right)_{(1,\,1)} = -1$$

Equation of tangent at (1, 1) is

$$y - 1 = -1\,(x - 1)$$

$$\Rightarrow \quad y + x - 2 = 0$$

Slope of normal = 1

Equation of normal is

$y - 1 = 1(x - 1) \Rightarrow y = x$

OR

$f(x)$ = sinx + cosx

$\Rightarrow$ $f(x) = \sqrt{2}\left(\sin x.\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos x\right)$

$f(x) = \sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$

$f'(x) = -\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

for $f(x)$ to be strictly increasing

$f'(x) > 0$

$\Rightarrow$ $-\sqrt{2}\sin\left(x-\frac{\pi}{4}\right) > 0$

$\Rightarrow$ $\sin\left(x-\frac{\pi}{4}\right) < 0$

$\Rightarrow$ $\pi < x - \frac{\pi}{4} < 2\pi$

$\Rightarrow$ $\pi+\frac{\pi}{4} < x < 2\pi+\frac{\pi}{4}$

$\Rightarrow$ $(2n-1)\pi+\frac{\pi}{4} < x < 2n\pi+\frac{\pi}{4}$

as $x \in [0, 2\pi]$

$\therefore$ $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

and for $f(x)$ to be decreasing

$f'(x) < 0$

$-\sqrt{2}\sin\left(x-\frac{\pi}{4}\right) < 0$

$\Rightarrow$ $\sin\left(x-\frac{\pi}{4}\right) > 0$

$\Rightarrow$ $0 < x - \frac{\pi}{4} < \pi$

$\Rightarrow$ $\frac{\pi}{4} < x < \pi+\frac{\pi}{4}$

$\Rightarrow$ $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

16. $\int \frac{1}{1-2\sin x}dx$

$$I = \int \frac{1}{1-2\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx$$

$$I = \int \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}-4\tan\frac{x}{2}}dx$$

Put $\tan\frac{x}{2} = t$

$\Rightarrow \sec^2\left(\frac{x}{2}\right)dx = 2dt$

$$I = \int \frac{2dt}{1+t^2-4t} = 2\int \frac{dt}{(t-2)^2-(\sqrt{3})^2}$$

$$I = 2\times\frac{1}{2\sqrt{3}}\log\left|\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right|+C$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{\tan\frac{x}{2}-2-\sqrt{3}}{\tan\frac{x}{2}-2+\sqrt{3}}\right|+C$$

OR

$$I = \int \frac{1}{\sqrt{3}\sin x+\cos x}dx$$

$$= \int \frac{1}{2\left(\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x\right)}dx$$

$$= \int \frac{1}{2\left(\cos\left(x-\frac{\pi}{3}\right)\right)}dx$$

$$= \frac{1}{2}\int \sec\left(x-\frac{\pi}{3}\right)dx$$

$$= \frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right|+C$$

17. $\Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$= (-1)^2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + (abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (using $c_3 \leftrightarrow c_2$, $c_1 \leftrightarrow c_2$)

$= (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$= (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$ (using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$= (1+abc)(b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$

$= (1 + abc)(b - a)(c - a)(c - b)$ (on expanding along c_1)

$\Delta = 0$ and a, b, c are different

$\therefore$ $a - b$, $b - c$, $c - a \neq 0$

$\Rightarrow$ $1 + abc = 0$

18. $I = \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x} dx$ ... (i)

$I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$

$I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2\cos^2 x + b^2\sin^2 x}$ ... (ii)

(i) + (ii)

$2I = \int_0^{\frac{2\pi}{2}} \frac{\pi}{a^2\cos^2 x + b^2\sin^2 x} dx$

$$2I = 2\pi\int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx$$

$$= 2\pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$$

Let us put $\tan x = t$, $\sec^2 x dx = dt$

when $x \to 0^+ \Rightarrow \tan x \to 0$

$x \to \frac{\pi}{2}- \Rightarrow \tan x \to \infty$

$$I = \pi\int_0^{\infty} \frac{dt}{a^2 + b^2t^2}$$

$$= \frac{\pi}{ab}\left[\tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}$$

$$= \frac{\pi}{ab}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi^2}{2ab}$$

OR

$$I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx \qquad \text{... (i)}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx \qquad \text{... (ii)}$$

(i) + (ii)

$$2I = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx$$

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Let $\cos x = t$, so that $-\sin x dx = dt$

at $x = 0$, $t = 1$, $x = \pi$, $t = -1$

$$I = \frac{\pi}{2}\int_{1}^{-1}\frac{-dt}{1+t^2}$$

$$= \frac{-\pi}{2}\left[\tan^{-1} t\right]_1^{-1}$$

$$= \frac{-\pi}{2}\left[\frac{-\pi}{4}-\frac{\pi}{4}\right] = \frac{\pi^2}{4}$$

19. Reflexivity : Let (a, b) *be* an arbitrary element of $N \times N$. Then,

$(a, b) \in N \times N$

$\Rightarrow a, b \in N$

$\Rightarrow a + b = b + a$

$\Rightarrow (a, b) R(a, b)$ for all $(a, b) \in N \times N$, So R is reflexive on $N \times N$

symmetry : Let $(a, b), (c, d) \in N \times N$ be such that $(a, b) R (c, d,)$

$(a, b) R (c, d,)$

$\Rightarrow a + d = b + c$

$\Rightarrow c + b = d + a$

$\Rightarrow (c, d) R (a, b)$

Thus $(a, b) R (c, d) \Rightarrow (c, d) R (a, b)$ for all $(a, b), (c, d) \in N \times N$

So, R is symmetric on $N \times N$.

Transitivity

Let $(a, b), (c, d), (e, f) \in N \times N$ such that

$(a, b) R (c, d)$ and $(c, d) R (e, f)$ then

$$\left.\begin{matrix}(a, b)R(c, d) \Rightarrow a+d = b+c \\ (c, d)R(e, f) \Rightarrow c+f = d+e\end{matrix}\right\} \Rightarrow (a+d)+(c+f) = (b+c)+(d+e)$$

$$\Rightarrow (a + f) = b + e$$

$$\Rightarrow (a, b) R (e, f)$$

Thus $(a, b) R (c, d)$ and $(c, d) R (e, f) \Rightarrow (a, b) R (e, f), (a, b), (c, d), (e, f) \in NxN$

So, R is transitive on $N \times N$

Hence, R is an equivalence relation.

20. We have

$\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a} \neq \vec{o}$

$\Rightarrow \vec{a}.\vec{b} - \vec{a}.\vec{c} = \vec{0}$ and $\vec{a} \neq \vec{o}$

$\Rightarrow$ $\vec{a}.(\vec{b}-\vec{c}) = 0$ and $\vec{a} \neq \vec{0}$

$\Rightarrow$ $\vec{b}-\vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b}-\vec{c})$

$\Rightarrow$ $\vec{b} = \vec{c}$ or $a \perp (\vec{b}-\vec{c})$

Again, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow$ $\vec{a}\times\vec{b} - \vec{a}\times\vec{c} = \vec{0}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow$ $\vec{a}\times(\vec{b}-\vec{c}) = \vec{0}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow$ $\vec{b}-\vec{c} = \vec{0}$ or $\vec{a} \,||\, (\vec{b}-\vec{c})$

$\Rightarrow$ $\vec{b} = \vec{c}$ or $\vec{a} \,||\, (\vec{b}-\vec{c})$

Combining the above two cases we get $\vec{b} = \vec{c}$.

21. Shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by

$$d = \left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

comparing the given equations with the equations $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ respectively

we get

$\vec{a}_1 = 4\hat{i} - \hat{j}$, $\vec{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b}_2 = 2\hat{i} + 4\hat{j} - 5\hat{k}$

Now $\vec{a}_2 - \vec{a}_1 = -3\hat{i} - 0\hat{j} + 2\hat{k}$

and $$\vec{b}_1\times\vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\hat{i} - \hat{j}$$

$\therefore$ $(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2) = (-3\hat{i} + 0\hat{j} + 2\hat{k}).(2\hat{i} - \hat{j})$

$= -6 + 0 + 0 = -6$

and, $|\vec{b}_1\times\vec{b}_2| = \sqrt{4+1+0} = \sqrt{5}$

$\therefore$ Shortest distance $= \left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right| = \left|\frac{-6}{\sqrt{5}}\right| = \frac{6}{\sqrt{5}}$

22. Let A_i be the event that ball drawn in i^{th} draw is white, where, $1 \le i \le 4$.

Since the balls are drawn with replacement. Therefore, A_1, A_2, A_3, A_4 are independent events such that

$P(A_i) = \frac{5}{20} = \frac{1}{4}$, $i = 1, 2, 3, 4$

Now required Probability

$= P(A_1 \cup A_2 \cup A_3 \cup A_4)$

$= 1 - P(\overline{A_1})P(\overline{A_2})P(\overline{A_3})P(\overline{A_4})$

$= 1 - \left(\frac{3}{4}\right)^4$

SECTION - C

23. $A = IA$

i.e., $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ (applying $R_1 \leftrightarrow R_2$)

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \to R_3 - 3R_1$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_1 \to R_1 - 2R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \to R_3 + 5R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$ (applying $R_3 \to \frac{1}{2}R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$ (applying $R_1 \to R_1 + R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$ (applying $R_2 \to R_2 - 2R_3$)

Hence $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

24. Let E_1, E_2, E_3, E_4 and A be the events as defined below

E_1 = the missing card is a heart card,

E_2 = the missing card is a spade card

E_3 = the missing card is a club card

E_4 = the missing card is a diamond card

A = Drawing two heart cards from the remaining cards.

Then,

$P(E_1) = \frac{13}{52} = \frac{1}{4}$

$P(E_2) = \frac{13}{52} = \frac{1}{4}$

$P(E_3) = \frac{13}{52} = \frac{1}{4}$

$P(E_4) = \frac{13}{52} = \frac{1}{4}$

$P\left(\frac{A}{E_1}\right)$ = Probability of drawing two heart cards given that one heart card is missing

$\Rightarrow \quad P\left(\frac{A}{E_1}\right) = \frac{^{12}C_2}{^{51}C_2}$

Similarly, $P\left(\frac{A}{E_2}\right) = \frac{^{13}C_2}{^{51}C_2}$

$P\left(\frac{A}{E_3}\right) = \frac{^{13}C_2}{^{51}C_2}$

$P\left(\frac{A}{E_4}\right) = \frac{^{13}C_2}{^{51}C_2}$

By Baye's theorem, we have,

Required probability = $P\left(\frac{E_1}{A}\right)$

$$= \frac{P(E_1).P\left(\frac{A}{E_1}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2).P\left(\frac{A}{E_2}\right)+P(E_3).P\left(\frac{A}{E_3}\right)+P(E_4).P\left(\frac{A}{E_4}\right)}$$

$$= \frac{\frac{1}{4}\times\frac{^{12}C_2}{^{51}C_2}}{\frac{1}{4}\times\frac{^{12}C_2}{^{51}C_2}+\frac{1}{4}\times\frac{^{13}C_2}{^{51}C_2}+\frac{1}{4}\times\frac{^{13}C_2}{^{51}C_2}+\frac{1}{4}\times\frac{^{13}C_2}{^{51}C_2}}$$

$$= \frac{^{12}C_2}{^{12}C_2+{}^{13}C_2+{}^{13}C_2+{}^{13}C_2} = \frac{11}{50}$$

OR

Let E_1, E_2, E_3 and A be the events defined as follows.

E_1 = the bolt is manufactured by machine A

E_2 = the bolt is manufactured by machine B

E_3 = the bolt is manufatured by machine C

A = the bolt is defective.

Then,

$P(E_1)$ = Probability that the bolt drawn is manufactured by machine A

$$= \frac{25}{100} = \frac{1}{4}$$

$P(E_2)$ = Probability that the bolt drawn is manufactured by machine B

$$= \frac{35}{100}$$

$P(E_3)$ = Probability that the bolt drawn is manufactured by machine C

$$= \frac{40}{100}$$

$P\left(\frac{A}{E_1}\right)$ = Probability that the bolt drawn is defective given that it is manufactured by A

$$= \frac{5}{100}$$

Similarly, $P\left(\frac{A}{E_2}\right) = \frac{4}{100}$

$$P\left(\frac{A}{E_3}\right) = \frac{2}{100}$$

Now

Required probability = Probability that the bolt is manufactured by machine B given that the bolt drawn is defective

$$= P\left(\frac{E_2}{A}\right)$$

$$= \frac{P(E_2).P\left(\frac{A}{E_2}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2).P\left(\frac{A}{E_2}\right)+P(E_3).P\left(\frac{A}{E_3}\right)}$$

$$= \frac{\frac{35}{100}\times\frac{4}{100}}{\frac{25}{100}\times\frac{5}{100}+\frac{35}{100}\times\frac{4}{100}+\frac{40}{100}\times\frac{2}{100}}$$

$$= \frac{28}{69}$$

25. Let x kg of food X and y kg of food Y are mixed together to make the mixture.

Amount of vitamin $A = x + 2y$ unit

Amount of vitamin $B = 2x + 2y$

Amount of vitamin $C = 3x + y$

According to requirement of vitamin A, B, C

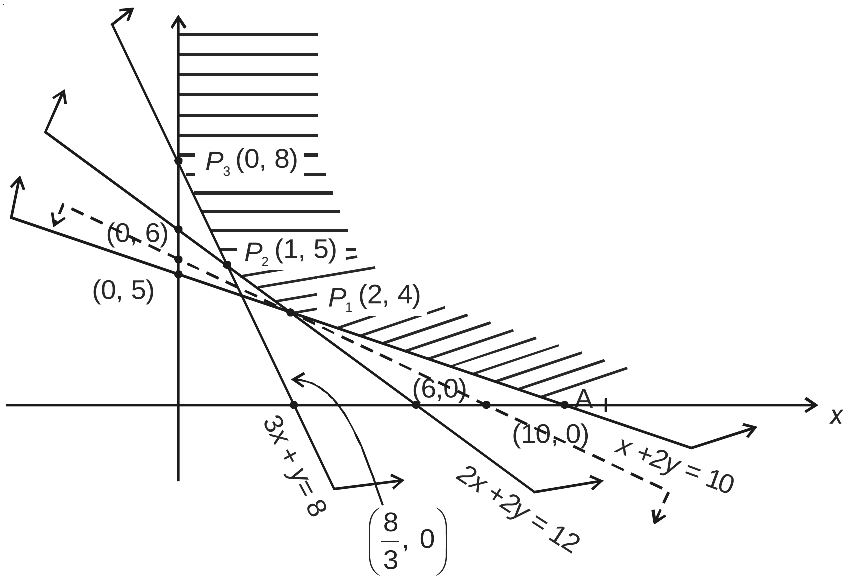
$x + 2y \geq 10$

$2x + 2y \geq 12$

$3x + y \geq 8$

and, $x \geq 0$, $y \geq 0$

We have to minimize $z = 6x + 10y$



The feasible region of *LPP* is shaded region

$A_1(10, 0)$, $P_1(2, 4)$, $P_2(1, 5)$ and $P_3(0, 8)$ are corner points.

Now, $z = 6x + 10y$ at $A_1(10, 0) = 6 \times 10 + 10 \times 0 = 60$

$z = 6x + 10y$ at $P_1(2, 4) = 6 \times 2 + 10 \times 4 = 52$

$z = 6x + 10y$ at $P_2(1, 5) = 6 \times 1 + 10 \times 5 = 56$

$z = 6x + 10y$ at $P_1(0, 8) = 6 \times 0 + 10 \times 8 = 80$

Clearly z is minimum at $x = 2$ and $y = 4$

The minimum value of z is 52

Now, let if possible $6x + 10y < 52$ and line does not enter in feasible region

Hence, minimum value of $6x + 10y = 52$

Hence, the least cost is Rs. 52.

26. Let M be the foot of the perpendicular from P on the plane $2x + 4y - z = 2$. Then PM is normal to the plane

So, its direction ratios are 2, 4, –1. Since PM passes through $P(7, 14, 5)$.

$\therefore$ Equation of PM is

$$\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = r(\text{say})$$

Let the co-ordinate of M be $(2r + 7, 4r + 14, -r + 5)$

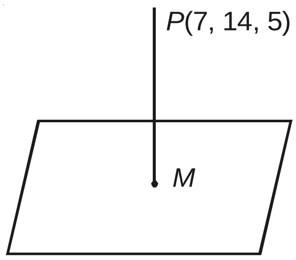
$\because$ M lies on the plane $2x + 4y - z = 2$

$\therefore$ $2(2r + 7) + 4(4r + 14) - (-r + 5) = 2$

$\Rightarrow$ $21r + 63 = 0$

$r = -3$

$\therefore$ Co-ordinates of M is (1, 2, 8)



PM = Length of perpendicular from P

$PM = \sqrt{(7-1)^2 + (14-2)^2 + (5-8)^2}$

$= 3\sqrt{21}$ unit

OR

Let Q be the image of the point $P(\hat{i} + 3\hat{j} + 4\hat{k})$ in the plane $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$. Then PQ is normal to the plane. Since PQ passes through P and is normal to the given plane, therefore equation of line PQ is

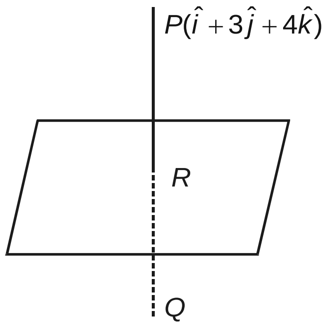
$\vec{r} = (\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$

$\because$ Q lies on line PQ so

The position vector of Q is

$= (\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$

$= (1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (4 + \lambda)\hat{k}$



$\because$ R is the mid point of PQ,

$\therefore$ Position vector of R is

$$\frac{[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(4+\lambda)\hat{k}]+(\hat{i}+3\hat{j}+4\hat{k})}{2}$$

$$= (\lambda+1)\hat{i}+\left(3-\frac{\lambda}{2}\right)\hat{j}+\left(4+\frac{\lambda}{2}\right)\hat{k}$$

$\because$ R lies on plane $\vec{r}.(2\hat{i}-\hat{j}+\hat{k})+3=0$

$\therefore$ $\left\{(\lambda+1)\hat{i}+\left(3-\frac{\lambda}{2}\right)\hat{j}+\left(4+\frac{\lambda}{2}\right)\hat{k}\right\}.(2\hat{i}-\hat{j}+\hat{k})+3=0$

$\Rightarrow$ $2\lambda+2-3+\frac{\lambda}{2}+4+\frac{\lambda}{2}+3=0$

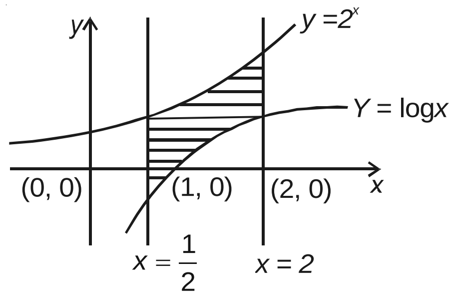
$\Rightarrow$ $\lambda = -2$

$\therefore$ Position vector of Q is $(\hat{i}+2\hat{j}+4\hat{k})-2(2\hat{i}-\hat{j}+\hat{k})$

$$= -3\hat{i}+4\hat{j}+3\hat{k}$$

27. Required area = area of shaded region

$$= \int_{\frac{1}{2}}^{2}(2^x-\log x)dx$$



$$A = \left(\frac{2^x}{\log 2}-x\log x+x\right]_{\frac{1}{2}}^{2}$$

$$= \left\{\frac{4}{\log 2}-2\log 2+2\right\}-\left\{\frac{\sqrt{2}}{\log 2}+\frac{1}{2}\log 2+\frac{1}{2}\right\}$$

$$A = \frac{(4-\sqrt{2})}{\log 2}-\frac{5}{2}\log 2+\frac{3}{2} \text{ sq. units}$$

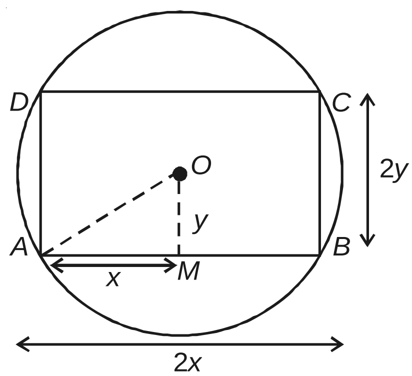
28. Let $ABCD$ be a rectangle in a given circle of radius a with centre at O.

Let $AB = 2x$, $AD = 2y$ be the sides of rectangle.

$AM^2 + OM^2 = OA^2$

$\Rightarrow$ $x^2 + y^2 = a^2$

$y = \sqrt{a^2-x^2}$ … (i)



Let P be the perimeter of the rectangle $ABCD$.

$P = 4x + 4y$

$\Rightarrow$ $P = 4x+4\sqrt{a^2-x^2}$

$\Rightarrow$ $\frac{dP}{dx} = 4-\frac{4x}{\sqrt{a^2-x^2}}$

for maximum value of P

$$\frac{dP}{dx} = 0$$

$$\Rightarrow \quad 4 - \frac{4x}{\sqrt{a^2 - x^2}} = 0$$

$$\Rightarrow \quad x = \frac{a}{\sqrt{2}}$$

Now, $$\frac{d^2P}{dx^2} = -\frac{4\left\{\sqrt{a^2 - x^2}\cdot 1 - \frac{x(-x)}{\sqrt{a^2 - x^2}}\right\}}{\left(\sqrt{a^2 - x^2}\right)^2} = \frac{-4a^2}{\left(a^2 - x^2\right)^{\frac{3}{2}}}$$

$$\therefore \left(\frac{d^2P}{dx^2}\right)_{x=\frac{a}{\sqrt{2}}} = -\frac{4a^2}{\left(a^2 - \frac{a^2}{2}\right)^{\frac{3}{2}}} = -\frac{8\sqrt{2}}{a} < 0$$

$\therefore$ P is maximum when $x = \frac{a}{\sqrt{2}}$

Putting $x = \frac{a}{\sqrt{2}}$ in (i)

$$y = \frac{a}{\sqrt{2}}$$

$$\therefore \quad x = y = \frac{a}{\sqrt{2}}$$

Hence, P is maximum when the rectangle is square of side $2x = \frac{2a}{\sqrt{2}} = \sqrt{2}a$

29. Given

$$\frac{dy}{dx} + (-2)y = \cos 3x \qquad \ldots \text{(i)}$$

Clearly it is linear differential equation

$$\text{I.F.} = e^{\int -2dx} = e^{-2x}$$

Multiplying both sides of (i) by I.F. = e^{-2x}

$$e^{-2x}\cdot\frac{dy}{dx} - 2ye^{-2x} = \cos 3x \cdot e^{-2x}$$

Integrating both sides w.r.t. x ,

$$ye^{-2x} = \int e^{-2x}\cos 3x dx + c$$

$ye^{-2x} = I + C$ … (ii)

Now, $I = \int e^{-2x}\cdot\cos 3x dx$

$$I = \frac{1}{3}e^{-2x}\cdot\sin 3x - \int\left(\frac{-2}{3}\right)e^{-2x}\sin 3x dx$$

$$I = \frac{1}{3}e^{-2x}\cdot\sin 3x + \frac{2}{3}\int e^{-2x}\sin 3x dx$$

$$I = \frac{1}{3}e^{-2x}\cdot\sin 3x + \frac{2}{3}\left[-\frac{1}{3}e^{-2x}\cdot\cos 3x - \frac{2}{3}\int e^{-2x}\cdot\cos 3x dx\right]$$

$$I = \frac{1}{3}e^{-2x}\cdot\sin 3x - \frac{2}{9}e^{-2x}\cdot\cos 3x - \frac{4}{9}I$$

$$\Rightarrow \left(I + \frac{4}{9}\right)I = \frac{e^{-2x}}{9}(3\sin 3x - 2\cos 3x)$$

$$\Rightarrow I = \frac{e^{-2x}}{13}(3\sin 3x - 2\cos 3x)$$

Putting the value of I in (ii)

$ye^{-2x} = \dfrac{e^{-2x}}{13}(3\sin 3x - 2\cos 3x) + C$ which is the required solution.

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