(1) Water seeps out of a conical filter at the constant rate of 5 cc / sec. When the height of water level in the cone is 15 cm, find the rate at which the height decreases. The filter is 20 cm high and the radius of its base is 10 cm.

[Ans: $\frac{4}{45\pi}$ cm / sec]

(2) A ship travels southwards at the speed of 24 km / hr and another ship 48 km to its south travels eastwards at 18 km / hr.
(a) Find the rate at which the distance between the two ships increases after 1 hour.
(b) Find the rate at which their distance increases after two hours.
(c) Explain the difference in the signs of the two rates.

[Ans: (a) - 8.4 km / hr, (b) 18 km / hr, (c) distance between the ships initially decreases till the ships are closest to each other. This happens at time, t = 1.28 hr when the rate of change of distance between them is zero.]

(3) The period T of simple pendulum of length l is given by the formula $T = 2\pi\sqrt{\frac{l}{g}}$. If the length is increased by 2 %, what is the approximate change in the period ?

[Ans: 1 % increase]

(4) Find the approximate value of $\sec^{-1}(-2\cdot01)$.

[Ans: $\frac{2\pi}{3} - \frac{1}{200\sqrt{3}}$]

(5) A formula for the amount of electric current passing through the tangent galvanometer is $i = k\tan\theta$, where θ is the variable and k is a constant. Prove that the relative error in i is minimum when $\theta = \frac{\pi}{4}$.

(6) Find the radian measure of the angle between the tangents to $y^2 = 4ax$ and $x^2 = 4ay$ at their point of intersection other than the origin.

[Ans : $\tan^{-1}\frac{3}{4}$]

(7) Prove that the portion of any tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ which lies between the co-ordinate axes is of constant length. (a > 0).

(8) If $\lambda_1 \neq \lambda_2$, then prove that the curves

$$\frac{x^2}{a^2 + \lambda_1} + \frac{y^2}{b^2 + \lambda_1} = 1 \quad \text{and} \quad \frac{x^2}{a^2 + \lambda_2} + \frac{y^2}{b^2 + \lambda_2} = 1$$

intersect each other orthogonally.

(9) Verify Rolle's Theorem for $f(x) = \sin x - \sin 2x$, $x \in [0, \pi]$.

(10) For $f(x) = x^3 - 6x^2 + ax + b$, it is given that $f(1) = f(3) = 0$. Find a and b and $x \in (1, 3)$ such that $f'(x) = 0$.

$$\left[\text{Ans: } a = 11, \; b = -6, \; x = 2 \pm \frac{1}{\sqrt{3}} \right]$$

(11) Apply mean value theorem to $f(x) = \log(1 + x)$ over the interval $[0, x]$ and prove that

$$0 < \frac{1}{\log(1 + x)} - \frac{1}{x} < 1, \quad (x > 0).$$

(12) Apply mean value theorem to $f(x) = e^x$ over the interval $[0, x]$ and prove that

$$0 < \log \frac{e^x - 1}{x} < x, \quad (x > 0).$$

(13) Prove that for $x > 0$, $\frac{x}{1 + x} < \log(1 + x) < x$.

(14) Length of each of the three sides of a trapezium is 5a. What should be the length of its fourth side if its area is maximum possible ?

[Ans: 10a]

(15) A 28 metre long wire is to be cut into pieces. One piece is to be bent to form a square and another piece is to be bent to form a circle. If the total area of the square and the circle has to be minimized, where should the wire be cut ?

$$\left[\text{Ans: } \frac{112}{\pi + 4} \text{ m for square and } \frac{28\pi}{\pi + 4} \text{ m for circle} \right]$$

(16) A window is in the shape of a semicircle over a rectangle. If the total perimeter of the window is to be kept constant and the maximum amount of light has to pass through the window, then prove that the length of the rectangle should be twice its height.

(17) The illumination due to an electric bulb at any point varies directly as the candlepower of the bulb and inversely as the square of the distance of the point from the bulb. Two electric bulbs of candlepowers C and 8C are placed 6 metres apart. At what point between them, the total illumination is minimum ?

[Ans: 2 m from the bulb having candlepower C]

(18) The perimeter of an isosceles triangle is constant and it is 100 cm. Its base increases at the rate of 0.2 cm / sec. Find the rate at which the altitude on the base increases, when the length of the base is 30 cm.

$$\left[\text{Ans: } -\frac{1}{2\sqrt{10}} \text{ cm / sec} \right]$$

(19) A boy flies a kite at a height of 50 m. The kite moves away from the boy at a horizontal velocity of 6 m / sec. Find the rate at which the string is released when the kite is 130 m away from the boy.

$$\left[\text{Ans: } \frac{72}{13} \text{ m / sec} \right]$$

(20) If a 5 % error is committed in the measurement of the radius of a sphere, what percentage of error will be committed in the calculation of its volume ?

[Ans: 15 %]

(21) If the error in measuring the radius of the base of a cone is δr and if its height is constant, what is the error committed in calculating the total surface area of the cone ?

$$\left[\text{Ans: } \left(2\pi r + \frac{\pi(2r^2 + h^2)}{\sqrt{r^2 + h^2}} \right) \delta r, \quad \left. \begin{array}{l} r = \text{radius of base} \\ h = \text{height of cone} \end{array} \right\} \right]$$

(22) In the calculation of the area of a triangle using the formula $\Delta = \frac{1}{2} bc \sin A$, A was taken as $\pi / 6$. Actually, an x % error crept into this measure of A. If b, c are constants, what is the percentage error in calculation of the area ? $\left[\text{Ans: } \frac{\sqrt{3}\,\pi x}{6} \% \right]$

(23) Prove that $y = x^3$ and $x^2 + 6y = 7$ intersect orthogonally.

(24) Find the equation of the tangent to $9x^2 + 4y^2 = 36$ which is perpendicular to the line $2x - 3y + 1 = 0$.

[Ans: $3x + 2y = \pm 6\sqrt{2}$]

(25) Find the measure of the angle between $xy = a^2$ and $x^2 + y^2 = 2a^2$.

[Ans: Angle is zero as both touch each other at (a, a) and $(-a, -a)$]

(26) Prove that $4x^2 + 9y^2 = 45$ and $x^2 - 4y^2 = 5$ intersect orthogonally.

(27) Determine whether Rolle's theorem is applicable in the following cases and if so, determine c such that $f'(c) = 0$.

(i) $f(x) = x + 3, \quad -2 \le x < 2$
$\quad\;\; = 7 - x, \quad 2 \le x < 6$

[Ans: f is not differentiable for $x = 2$, $f'(x) \ne 0$ for any $x \in (-2, 6)$]

(ii) $f(x) = 2x + 3, \quad x < 3$
$\quad\;\; = 15 - 2x, \quad x \ge 3, \quad x \in [1, 5]$

[Ans: f is not differentiable for $x = 3$, $f'(x) \ne 0$ for any $x \in (1, 5)$]

(iii) $f(x) = |x|, \quad x \in [-1, 1]$

[Ans: f is not differentiable for $x = 0$, $f'(x) \ne 0$ for any $x \in (-1, 1)$]

(28) Apply mean value theorem to $f(x) = ax^3 + bx^2 + cx + d$ on $[0, 1]$, $a, b > 0$.

$$\left[\text{Ans : } \frac{-b + \sqrt{3a^2 + 3ab + b^2}}{3a} \right]$$

(29) Find where $x^4 - 5x^3 + 9x^2 - 7x + 2$ is increasing and where it is decreasing.

$$\left[\text{Ans : f increases in } \left\{ x \mid x < \frac{7}{4},\ x \in R \right\}, \text{ f increases in } \left\{ x \mid x > \frac{7}{4},\ x \in R \right\} \right]$$

(30) Prove that $e^x > 1 + x$, $x \in R - \{0\}$.

(31) Prove that $\frac{\tan x}{x}$ is increasing over $\left(0, \frac{\pi}{2}\right)$.

(32) If $x > 0$, prove that $\log(1 + x) > x - \frac{x^2}{2}$.

Check for global or local extreme values: (33 to 37)

(33) $f(x) = 3x^4 - 10x^3 + 6x^2 + 5$, $x \in [-3, 3]$.

[Ans: global maxm. $f(-3) = 572$, global min. $f(2) = -3$,
local min. $f(0) = 5$ and $f(2) = -3$, local maxm. $f\left(\frac{1}{2}\right) = \frac{87}{16}$]

(34) $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 2$, $x \in [-2, 2]$.

[Ans: $f(-1) = \frac{19}{6}$ is a local and global maximum, $f'(2) = 0$, but $2 \notin (-2, 2)$,
$f(2) = -\frac{4}{3}$ is a global minimum.]

(35) $f(x) = x^{50} - x^{20}$, $x \in [0, 1]$.

[Ans: global maximum $f(0) = 0$. For $x = \left(\frac{2}{5}\right)^{\frac{1}{30}}$,
local minimum is $f(x) = \left(\frac{2}{5}\right)^{\frac{5}{3}} - \left(\frac{2}{5}\right)^{\frac{2}{3}}$]

(36) $f(x) = \sin x + \cos x$, $x \in R$.

[Ans: Max. $f\left(2k\pi + \frac{\pi}{4}\right) = \sqrt{2}$,
Min. $f\left(2k\pi + \frac{5\pi}{4}\right) = -\sqrt{2}$]

(37) $f(x) = x + \sin 2x, \quad x \in [0, \pi]$.

[Ans: local max. $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$, local min. $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$, global min. $f(0) = 0$, global max. $f(\pi) = \pi$.]

(38) If the length of the hypotenuse of a right triangle is given, prove that its area is maximum when it is an isosceles right triangle.

(39) If the volume of a right circular cone with given oblique height is maximum, prove that the radian measure of its semi vertical angle is $\tan^{-1}\sqrt{2}$.

(40) Find approximate value of $\log_{10} 999$. [Ans: 2.9995657]

(41) Prove that the length of the tangent is constant for the curve $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a \sin t$.

(42) Prove that for the curve $xy = a^2$, the mid-point of the segment of a tangent intercepted between the two axes is precisely the point of contact (i.e., the point of tangency).

(43) Using mean value theorem, prove that if $x > 0$, then $\frac{x}{1 + x^2} < \tan^{-1} x < x$.

(44) Prove that if $x > 0$, then $\frac{\log(1 + x)}{x}$ is a decreasing function.

(45) By cutting equal squares from the four corners of a 16 x 10 tin sheet, a box is to be constructed. What should be the length of each square if the volume of the box is to be maximum ?

[Ans: 2]

(46) Prove that x^x is minimum when $x = \frac{1}{e}$.

(47) The cost of manufacturing x TV sets per day is $\frac{1}{4}x^2 + 35x + 25$ and the sale price of one TV set then is $50 - \frac{1}{2}x$. How many TV sets should be manufactured to maximize profit ? Prove that then the cost of making one TV set is minimum.

(48) Find the minimum distance of (4, 2) from $y^2 = 8x$. [Ans: $2\sqrt{2}$]

(49) Which line passing through (3, 4) makes a triangle of minimum area with the two axes in the first quadrant ?

[Ans: $4x + 3y = 24$]

(50) The equation of motion of a particle moving on a line is $s = t^3 + at^2 + bt + c$, where displacement s is in m, time t is in sec., and a, b, c, are constants. $t = 1 \Rightarrow s = 7$ and velocity, $v = 7$ m/s $\Rightarrow$ acceleration, $a = 12\ m/s^2$. Find a, b, c.

[Ans: a = 3, b = - 2, c = 5]

(51) Find the rate of increase of the volume, V, of a sphere with respect to its surface area, S. $\left[\text{Ans: } \frac{dV}{dS} = \frac{1}{4}\sqrt{S/\pi} \right]$

(52) Find the area of an equilateral triangle with respect to its perimeter, p.

$\left[\text{Ans: } p/6\sqrt{3} \right]$

(53) A 2 m tall man walks towards a source of light situated at the top of a pole 8 m high, at the rate of 1.5 m / s. How fast is the length of his shadow changing when he is 5 m away from the pole ?

[Ans: - 2 m / s]

(54) Two sides of a triangle are 15 and 20. How fast is the third side increasing when the angle between the given sides is 60° and is increasing at the rate of 2° per sec. ?

$\left[\text{Ans: } \dfrac{\pi}{\sqrt{39}} \text{ unit / s} \right]$

(55) A metal ball of radius 90 cm. is coated with a uniformly thick layer of ice, which is melting at the rate of 8π cc / min. Find the rate at which the thickness of the ice is decreasing when the ice is 10 cm thick.

[Ans: 0.0002 cm / min]

(56) Obtain the equations of the tangent and the normal to the curve $x = a\cos\theta$ and $y = b\sin\theta$ at θ-point.

$\left[\text{Ans: } \dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1, \quad \dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2 \right]$

(57) Obtain the equations of the tangent and the normal to the curve $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2$ at a point where $x = a$.

$\left[\text{Ans: } \dfrac{x}{a} + \dfrac{y}{b} = 2, \quad ax - by = a^2 - b^2 \right]$

(58) Obtain the equations of the tangent and the normal to the curve $x = 2e^t$, $y = e^t$ at $t = 0$.

[Ans: $x - 2y = 0, \quad 2x + y - 5 = 0$]

(59) Find the lengths of the sub-tangent, sub-normal, tangent and normal at $\theta = \dfrac{\pi}{3}$ to the curve $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$.

$\left[\text{Ans: } \dfrac{\sqrt{3}}{2}|a|, \quad \dfrac{|a|}{2\sqrt{3}}, \quad |a|, \quad \dfrac{|a|}{\sqrt{3}} \right]$

(60) For $f(x) = (x - a)^m (x - b)^n$, $x \in [a, b]$ and $m, n \in N$, determine whether Rolles' theorem is applicable and if so, determine $c \in (a, b)$ such that $f'(c) = 0$.

$\left[\text{Ans: Rolles' theorem is applicable and } c = \dfrac{mb + na}{m + n} \right]$

(61) Apply the mean value theorem to $f(x) = \cos x$, $x \in \left[0, \dfrac{\pi}{2}\right]$ and prove that $\tan \dfrac{x}{2} < x$.

(62) Show that $0 < x_1 < x_2 < \dfrac{\pi}{2} \Rightarrow x_1 - \sin x_1 < x_2 - \sin x_2$.

(63) For $x > 0$, prove that $x - \dfrac{x^3}{3} < \tan^{-1} x < x$.

(64) Using $f(x) = x^{\frac{1}{x}}$, $x > 0$, which is greater, e^{π} or π^{e} ? [Ans: e^{π}]

(65) Apply Mean value theorem to $f(x) = \log(1 + x)$ over the interval $[a, b]$, where $0 < a < b$ and prove that $\dfrac{b - a}{b} < \log\left(\dfrac{b}{a}\right) < \dfrac{b - a}{a}$.

(66) Using $f(x) = \tan x \sin x - x^2$, $0 < x < \dfrac{\pi}{2}$, prove that $\dfrac{x}{\sin x} < \dfrac{\tan x}{x}$.

(67) Prove that the rectangle inscribed in a given semi-circle has maximum area, if its length is double its breadth.

(68) Fuel cost of running a train is proportional to square of its speed in km / hr. and is Rs. 100 / hr at 10 km / hr. What is its most economical speed, if the fixed charges are Rs. 400 per hour ?

[Ans: 20 km / hr]