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# MATHEMATICS

*Duration: Three Hours* *[Maximum Marks: 150*

- **Read the following instructions carefully**
  1. Write all the answers in the answer book.
  2. This question paper consists of TWO SECTIONS: A and B.
  3. Section A has Seven questions. Answer All questions in this section.
  4. Section B has Twenty questions. Answer any TEN questions from this section. If more number of questions is attempted, strike off the answers not to be evaluated; else only the First Ten un scored answers will be considered.
  5. Answers to Section B should start on a fresh page and should not be mixed with answers to Section A.
  6. Answers to questions and answers to parts of a question should appear together and should not be separated.
  7. In all questions of 5 marks, write clearly the important steps in your answer. These steps carry partial credit.
  8. There will be no negative marking.
- **Note**
  1. The symbols N, Z, Q, R and C denote the set of natural numbers, integers, rational numbers, real numbers, and complex numbers respectively.
  2. Zn/nZ denotes the ring of congruence classes modulo n.
  3. $C^1[a,b]$ Denotes the space of continuously differentiable functions.

## SECTION-A (100 MARKS)

1. This question has 30 parts. Each part carries two marks. For each part only one of the suggested alternatives is correct. Write the alphabet corresponding to the correct alternative in your answer book.

(30×2 = 60)

1.1 Let M be a $mXn(m<n)$ matrix with rank $m$. Then

(A) For every b in $R^m_, Mx=b$ has unique solution

(B) For every b in $R^m_, Mx=b$ has a solution but it is not unique

(C) There exists $b\in R^m$ for which $Mx=b$ has not solution

(D) None of the above

1.2 Let $M=\begin{bmatrix}1&1&0\\-1&1&2\\2&2&0\\-1&0&1\end{bmatrix}$. Then the rank of M is equal to

(A) 3

(B) 4

(C) 2

(D) 1

1.3 Let F be a finite field. If $f:F\to F$, given by $f(x)=x^3$ is a ring homomorphism, then

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(A) F=Z/3Z

(B) F=Z/2Z or characteristic of F = 3

(C) F=Z/2Z or Z/3Z

(D) Characteristic F is 3

1.4 If S is a finite commutative ring with 1 then

(A) Each prime ideal is a maximum ideal

(B) S may have a prime ideal which is not maximal

(C) S has no nontrivial maximal ideals

(D) S is a field

1.5 Let f be an entire function. If f satisfies the following two equations $f(z+1)=f(z)$,

$f(z+1)=f(z)$ for every z in C, then

(A) $f'(z)=f(z)$

(B) $f(z)\in R\forall_z$

(C) $\equiv$ constant

(D) f is a non-constant polynomial

1.6 The residue of $\frac{\sin z}{z^8}$ at z = 0 is

(A) 0

(B) $-\frac{1}{71}$

(C) $\frac{1}{71}$

(D) None of these

1.6.1 Let $\Gamma$ denote the boundary of the square whose sides lies along $x=\pm 1$ and $y=\pm 1$, where $\Gamma$ is described in the positive sense. Then the value of $\int_\Gamma \frac{z^2}{2z+3}dz$ is

(A) $\frac{\pi i}{4}$

(B) $2\pi i$

(C) 0

(D) $-2\pi i$

1.7 Let X be the space of polynomials in one variable

For $p\in X$, $p(t)=a_0+a_1t+...+a_nt^n$, let

$\|p\|=Sub\{|p(t)|:0\le t\le 1\}$,

$\|p\|_1=|a_0|+|a_1|+...+|a_n|$

Then

(A) X is complete with $\|.\|$

(B) X is complete with $\|.\|_1$

(C) X cannot be nor med so as to make it complete

(D) None of the above

1.8 Let X and Y be normal linear spaces. Every linear map $T: X \to Y$ is continuous if

(A) $\dim(X) < \infty$

(B) $\dim(Y) < \infty$

(C) $\dim(Y) = 1$

(D) $X = l^2$

1.9 All the eigen values of the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ lie in the disc

(A) $|\lambda+1| \le 1$

(B) $|\lambda-1| \le 1$

(C) $|\lambda+1| \le 0$

(D) $|\lambda-1| \le 2$

1.10 $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$ is equal to

(A) $x^2$

(B) 1

(C) $x$

(D) $x^n$

1.11 The equation $e^x - 4x^2 = 0$ has a root between 4 and 5. Fixed point iteration with iteration function $\frac{1}{2}e^{x/2}$

(A) Diverges

(B) Converges

(C) Oscillates

(D) Converges monotonically

1.12 The formula $A_0 f\left(-\frac{1}{2}\right) + A_2 f(0) + A_2 f\left(\frac{1}{2}\right)$ which approximates the integral $\int_{-1}^{1} f(x)dx$ is exact for polynomials of degree less than or equal to 2 if

(A) $A_0 = A_2 = \frac{4}{3}, A_1 = \frac{2}{3}$

(B) $A_0 = A_1 = A_2 = 1$

(C) $A_0 = A_2 = \frac{4}{3}, A_1 = -\frac{2}{3}$

(D) None of the above

1.13 If $x=\xi$ is a double root of the equation $f(x)=0$ and if $f''(x)$ is continuous in a neighborhood of $\xi$ then the iteration scheme for determining $\xi$: $x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)}$ has the order

(A) 2
(B) 1
(C) Less than 2
(D) Less than 1

1.14 Let $f(x,y)=\frac{xy}{x^2+y^2},(x,y)\neq(0,0);\ f(0,0)=0$. Then

(A) f is continuous at (0,0) and the partial derivatives $f_{x'} f_y$ exist at every point of $R^2$

(B) f is discontinuous at (0,0) and $f_{x'}$ f exist at every point of $R^2$

(C) f is discontinuous at (0,0) and $f_{x'} f$ exist only at (0,0)

(D) None of the above

1.15 Let $f:[a,b]\to(0,\infty)$ be continuous. Let $G_f=\{(x,y):y=f(x)\}$ be the graph of f. Then

(A) $G_f$ is measurable with measure zero in $R^2$
(B) $G_f$ is measurable only if f is differentiable in (a, b)
(C) $G_f$ is measurable and the measure of $G_f$ lies between $(b-a)f(a)$ and $(b-a)f(b)$
(D) $G_f$ need not be measurable

1.16 Let $f:R\to R$ be a continuous function. If $f(Q)\subseteq N$, then

(A) $f(R)=N$

(B) $f(R)\subseteq N$ but f need not be constant

(C) f is unbounded
(D) f is a constant function

1.17 Let $f:[a,b]\to R$ be a monotonic function. Then

(A) f is continuous
(B) f is discontinuous at most two points
(C) f is discontinuous at finitely many points
(D) f is discontinuous at most countable points

1.18 Let $f:[0,10)\to[0,10]$ be a continuous mapping. Then

(A) f need not have any fixed point
(B) f has at least 10 fixed points
(C) f has at least 9 fixed points
(D) f has at least one fixed point

1.19 Let X be a nonempty set and $\tau_1,\tau_2$ be two topologies on X. Let $f:(X,\tau_1)\to(X,\tau_2)$ be a mapping. If f is a homomorphism, then

(A) $\tau_1=\tau_2$

(B) $\tau_1=\tau_2$ and f is the identity map

(C) $\tau_1$ is finer than $\tau_2$ implies X is infinite

(D) Either $\tau_1$ is finer than $\tau_2$ or $\tau_2$ is finer than $\tau_1$

1.20 The orthogonal trajectories of the family $x^2 - y^2 = C_1$ are given by

(A) $x^2 + y^2 = C_2$

(B) $xy = C_2$

(C) $y = C_2$

(D) $x = C_2$

1.21 For the differential equation $t(t-2)^2 y'' + ty' + y = 0, \quad t = 0$ is

(A) An ordinary point

(B) A branch point

(C) An irregular point

(D) A regular singular point

1.22 Suppose that $y_1$ and $y_2$ form a fundamental set of solutions of a second order ordinary differential equation on the interval $-\infty < t < +\infty$, then

(A) There is only one zero of $y_1$ between consecutive zeros of $y_2$

(B) There are two zeros of $y_1$ between consecutive zeros of $y_2$

(C) There is finite number of zeros of $y_1$ between consecutive zeros of $y_2$

(D) None of the above

1.23 The general solution of the system of differential equations $\frac{dX}{dt} = MX + b$ where $X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; M$, a $2\times 2$ matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and b, a $2\times 1$ constant vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is given by

(A) $e^{Mt}c + b$

(B) $e^{Mt}c + bt$

(C) $e^{Mt}c - b$

(D) $e^{Mt}c - bt$ where c is any $2 \times 1$ constant vector.

1.24 The general solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is of the form

(A) $u = f(x+iy) + g(x-iy)$

(B) $u = f(x+y) + g(x-y)$

(C) $u = cf(x-iy)$

(D) $u = g(x+iy)$

1.25 Suppose P and Q are independent events of an experiment E. Then it follows that

(A) P and Q are dependent

(B) P and Q are mutually exclusive



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(C) P and Q are dependent

(D) P and Q are independent

1.26 Let $X_n$ denote the sum of points obtained when $n$ fair dice are rolled together. The expectation and variance of $X_n$ are

(A) $\frac{7}{2}n$ and $\frac{35}{12}n^2$ respectively

(B) $\frac{7}{2}n$ and $\frac{35}{12}n$ respectively

(C) $\left(\frac{7}{2}\right)^n$ and $\left(\frac{35}{12}\right)^n$ respectively

(D) None of the above

1.27 Let $b_{yx}$ and $b_{xy}$ denote the regression coefficient of Y on X and of X on Y respectively. Then $b_{yx} = b_{xy}$ implies that

(A) $\sigma_y = \sigma_x$

(B) $\rho = 1$

(C) $\rho = 0$

(D) None of the above

1.28 The test of a simple null hypothesis $H_0 : \theta = \theta_0$ of size α is biased if

(A) The power function of the test assumes a value less than α for some $\theta \neq \theta_0$

(B) The power function has a value less than α when $\theta = \theta_0$

(C) The probability of accepting the null hypothesis when it is true is larger than the probability of rejecting the null hypothesis when it is true

(D) The power function has a value greater than α when $\theta = \theta_0$

1.29 Consider the following LPP:

Max $z = x_1$

Subject to $x_1 + x_2 \leq 1$

$x_1 - x_2 \geq 1$

$-2x_1 + x_2 \geq 1$

$x_1 \geq 0$

And $x_2 \geq 0$

The solution to its dual is

(A) Unbounded

(B) Infeasible

(C) Degenerate

(D) Bounded

2. This question has 3 parts. Each part has two column of several entries. Each entry in the left column corresponds to a unique entry in the right column. Write the correct pairs in your answer book.

(5×3 = 15)



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2.1 Let V be a finite dimensional vector space over R

(i) There exists a linear map $f:V\rightarrow$ such that $f^2=-Id$

(ii) For all linear maps $f:V\rightarrow V$ there exists a proper 1–dim.

(iii) V is finite union of 1-dim. Subspaces

(iv) There exists a unique $f:V\rightarrow V$ such that $f^2=f$

(v) There exists a proper 1-dim. Subspace W such that V\W is connected

(A) $\dim(V)\geq 3$

(B) $\dim(V)=1$

(C) dim (V) is odd

(D) dim (V) is even

(E) dim (V) = 0

2.2

(i) $l^1$

(ii) $\left(C^1[a,b],\|.\|_\infty\right)$

(iii) $l^2$

(iv) $l^\infty$

(v) $\left(R^n,\|.\|_1\right)$

(A) Norm satisfies parallelogram law

(B) Complete, non-reflexive, separable

(C) Non-separable, complete

(D) Finite basis

(E) Not complete

(F) Denumerable (Hamel) basis

2.3 In a series of Bernoulli trials with probability p of success the appropriate distribution to obtain probability that

(i) 1st success occurs at the xth trial is

(ii) $k^{th}(k>1)$ success occurs at the xth trial is

(iii) $x$ number of failures occur in order to register $k(>1)$ number of success is

(iv) x number of successes occur in k trials is

(v) x number of successes occur in k trials for $k\rightarrow\infty$ and $p\rightarrow 0$ such that $kp\rightarrow\alpha$ is

(A) $\binom{x-1}{k-1}p^k q^{x-k}$

(B) $\binom{k}{x}p^x q^{k-r}$

(C) $\binom{x+k-1}{k-1}p^k q^x$

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(D) $e^{-\alpha}\dfrac{\alpha^x}{x!}$

(E) $q^{x-1}p$

(F) $\alpha e^{-\alpha x}$

3. Let A be a $n\times n$ normal matrix. Show that $Ax=\lambda x$ if and only if $A^*x=\bar{\lambda}x$. If A has, in addition, n distinct real eigen values, show that A is hermitian.

(5)

4. Let $\{u_n : n=1,2,......\}$ be an orthonormal set in a Hilbert space and $x\in H$. Show that $\sum_{n=1}^{\infty}\left|\langle x,u_n\rangle\right|^2=\|x\|^2$, if and only if $x\in$ span $(u_n : n=1,2,.....)$.

(5)

5. Let $\sum_{n=1}^{\infty}a_n$ be a convergent series with $a_n\geq 0$ for all n and let $f:R\to R$ be a continuous function. Show that the series $\sum_{n=1}^{\infty}a_n f(a_n x)$ is convergent for all x.

(5)

6. Derive the transcendental equation for determining the eigen values $\lambda$ of the BVP:

$y''+\lambda^2 y=0,\qquad y(0)=0,\qquad y(1)=y'(1)$

Determine the smallest positive eigen value correct to two decimal places.

(5)

7. The joint probability density function of X and Y is given by

$$f(x,y)=\begin{cases}\dfrac{4(x+y)}{5x^3}, & 0\leq y\leq 1, 1\leq x\leq\infty\\ 0 & \text{Otherwise}\end{cases}$$

Let $g(A)=P\left(0<Y<\frac{1}{2}\middle| X>A\right)$

Find the maximum and the minimum value of g(A).

(5)

## SECTION B (50 Marks)

***Answer any TEN questions from this section. Each question carries 5 Marks.***

8. Let $f : R^3 \to R^3$ be a linear mapping. Show that there exists a one dimensional subspace V or $R^3$ such that it is invariant under f.

9. Let G be a subgroup of the multiplication group of all nonzero complex numbers. If G is finite, then show that G is cyclic.

10. If f is a non-constant entire function, then show that the range of f is dense in C.

11. Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2+b^2}}, a > b > 0.$

12. Let X be a Banach space, Y and Z closed subspaces of X such that $X = Y + Z$ and $Y \cap Z = \{0\}$. If $P : X \to Y$ is a map given by $P(x) = y$, where $x + z,\ y \in Y$ and $z \in Z$, then show that P is continuous.

13. Let H be a Hilbert space and F a closed subspace of H. Let $a \in H$ and $a \notin F$. Show that there exists a linear functional f on H such that

    (i) $f \equiv 0$ on F

    (ii) $f(a) = d(a, F)$, the distance of a from F and

    (iii) $\|f\| = 1$

14. Given the BVP: $y'' + y = 0,\ y'(0) + y(0),\ y(1)$, use the second order finite difference method with the step size h = 0.2 and set up in the matrix form the system of 5 equations in 5 unknowns.

15. The function defined by $f(x) = \int_1^x \frac{dt}{2\sqrt{t}}$ is tabulated for equally spaced value of x with h = 0.1. What is the maximum error encountered if piecewise quadratic interpolation is to be used to calculate $f(\overline{x})$ where $\overline{x} \in [1, 2]$?

16. Suppose that $f(x) \geq 0$ for all x in $[0,1]\, f$ continuous on $[0,1]$ and $\int_0^1 f(x)\,dx = 0$. Show that $f \equiv 0$.

    If g is continuous on $[0,1]$ and $\int_0^1 g(x)\,x^n dx = 0; n = 0, 1, 2, \ldots$ then prove also that $g \equiv 0$.

17. Let $f : R^2 \to R$ be a function such that $|\, f(x, y) \leq x^2 + y^2$ for all $(x, y) \in R^2$. Show that f is differentiable at (0,0).

18. Using Lap lace transform, solve

    $$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y = e^{2t},\ y(0) = 1; y'(0) = -1$$

19. Find the complete integral of the equation



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$\left(p^2+q^2\right)x=pz$, where $p=\dfrac{\partial z}{\partial x}, q=\dfrac{\partial z}{\partial y}$

20. Solve $\dfrac{\partial\theta}{\partial t}=\dfrac{\partial^2\theta}{\partial x^2}, 0\le x\le a, t>0$ subject to the conditions:

$\theta(0,t)=\theta(a,t)=0$, and $\theta(x,0)=\theta_0$ (Constant )

21. A string of length $l$ and mass $\mu$ per unit length, with fixed ends, is stretched to a tension F. Derive the equations governing the displacement $y(x,t)$ using Lagrange's equations.

22. Let $S^1$ be the circle in $R^2$ with the subspace topology. Show that there does not exist any injective continuous function $f:S^1\to R$.

23. Let X be a count ably infinite set. Construct a topology $\tau$ on X such that $(X,\tau)$ is Haudorff and $\tau$ is not discrete.

24. (a) If T is an unbiased estimator of θ, then show that $T^2$ is not necessarily an unbiased estimator of $\theta^2$.

(b) Given that $T_1$ and $T_2$ are both efficient estimators of θ with variance v, find the condition so that $T=\dfrac{T_1+T_2}{2}$ is also an efficient estimator of θ.

25. Let $Y_1, Y_2, \ldots\ldots, Y_{15}$ be a random sample of size 15 from the probability density function $f_y(y)=3(1-y)^2, 0<y<1$

Use the central limit theorem to approximate. $P\left(\dfrac{1}{8}<\bar{Y}<\dfrac{3}{8}\right)$.

26. Suppose that $\tilde{X}=\left(X_1, X_2, \ldots\ldots; X_{100}\right)$ is a random sample from a normal distribution with mean θ and standard deviation 1.8. it is desired to test the null hypothesis $H_0:\theta=2$ against $H_1:\theta\ne 2$, using a significance level 0.05. Derive the power function of the two-tailed test with critical region $\left\{\tilde{X}:\left|\bar{X}-2\right|\ge k\right\}$. Give a rough sketch of the power function.

27. A transportation problem for which the costs, origin and availabilities, destination and requirements are given as follows:

| | $D_1$ | $D_2$ | $D_3$ | |
|---|---|---|---|---|
| $Q_1$ | 2 | 1 | 2 | 40 |
| $Q_2$ | 9 | 4 | 7 | 60 |
| $Q_3$ | 1 | 2 | 9 | 10 |
| | 40 | 50 | 20 | |

Check whether the following basic feasible solution

$x_{11}=20, x_{13}=20, x_{21}=10, x_{22}=50$

$x_{31}=10$ and $x_{12}=x_{23}=x_{32}=x_{33}=0$

Is optimal, If not, find an optimal solution