MATHEMATICS

Duration: Three Hours ***Maximum Marks: 150***

Notations and Definitions used in the paper

R: The set of real numbers.

$R^n = \{(x_1, x_2, ..., x_n) : x_i \in R,\ i = 1, 2, ..., n\}$

C: The set of complex numbers.

ϕ : The empty set.

For any subset E of X (or a topological space X).

$\bar{E}$: The closure of E in X.

E°: The interior of E in X.

E^c : The complement of E in X.

$Z_n = \{0, 1, 2, ..., n-1\}$

A^t : The transpose of a matrix A.

ONE MARKS QUESTIONS (1-20)

1. Consider R2 with the usual topology. Let $S = \{(x, y) \in R^2 : x \text{ is an integer}\}$. Then S is

   (a.) Open but Not Closed

   (b.) Both open and closed

   (c.) Neither open nor closed

   (d.) Closed but Not open

2. Suppose $X = \{\alpha, \beta, \delta\}$. Let

   $\Im_1 = \{\phi, X, \{\alpha\}, \{\alpha, \beta\}\}$ and $\Im_2 = \{\phi, X, \{\alpha\}, \{\beta, \delta\}\}$.

   Then

   (a.) Both $\Im_1 \cap \Im_2$ and $\Im_1 \cup \Im_2$ are topologies

   (b.) Neither $\Im_1 \cap \Im_2$ nor $\Im_1 \cup \Im_2$ is a topology

   (c.) $\Im_1 \cup \Im_2$ is a topology but $\Im_1 \cap \Im_2$ is Not a topology

   (d.) $\Im_1 \cap \Im_2$ is a topology but $\Im_1 \cup \Im_2$ is not a topology

3. For a positive integer n, let $f_n : R \to R$ be defined by

$$f_n(x)=\begin{cases}\frac{1}{4n+5}, & \text{If } 0\le x\le n\\ 0 & \text{Otherwise}\end{cases}$$

Then $\{f_n(x)\}$ converges to zero

(a.) Uniformly but Not in L^1 norm

(b.) Uniformly and also in L^1 norm

(c.) Point wise but Not uniformly

(d.) In L^1 norm but Not point wise

4. Let P_1 and P_2 be two projection operators on a vector space. Then

(a.) P_1+P_2 is a projection if $P_1P_2=P_2P_1=0$

(b.) P_1-P_2 is a projection if $P_1P_2=P_2P_1=0$

(c.) P_1+P_2 is a projection

(d.) P_1-P_2 is a projection

5. Consider the system of linear equations

$x+y+z=3,\ x-y-z=4,\ x-5y+kz=6$

Then the value of k which this system has an infinite number of solutions is

(a.) k = –5

(b.) k = 0

(c.) k = 1

(d.) k = 3

6. Let $A=\begin{bmatrix}1&1&1\\2&2&3\\x&y&z\end{bmatrix}$ and let $V=\{(x,y,z)\in R^3:\det(A)=0\}$. Then the dimension of V equals

(a.) 0

(b.) 1

(c.) 2

(d.) 3

7. Let $S=\{0\}\cup\left\{\frac{1}{4n+7}:n=1,2,...\right\}$. Then the number of analytic functions which banish only on S is

(a.) Infinite

(b.) 0

(c.) 1

(d.) 2

8. It is given that $\sum_{n-0}^{\infty} a_n z^n$ converges at z = 3+i4. Then the radius of convergence of the power series $\sum_{n-0}^{\infty} a_n z^n$ is

(a.) ≤ 5

(b.) ≥ 5

(c.) <5

(d.) >5

9. The value of α for which $G = \{\alpha, 1, 3, 9, 19, 27\}$ is a cyclic group under multiplication modulo 56 is

(a.) 5

(b.) 15

(c.) 25

(d.) 35

10. Consider Z_{24} as the additive group modulo 24. Then the number of elements of order 8 in the group Z_{24} is

(a.) 2

(b.) 2

(c.) 3

(d.) 4

11. Define $f: R^2 \to R$ by $f(x, y) = \begin{cases} 1, & if \;\; xy = 0, \\ 2, & otherwise \end{cases}$

If $S = \{(x, y): f$ is continuous at the point $(x, y)\}$, then

(a.) S is open

(b.) S is closed

(c.) $S = \phi$

(d.) S is closed

12. Consider the linear programming problem,

$\max. z = c_1x_1 + c_2x_2, c_1, c_2 > 0$

Subject to. $x_1 + x_2 \leq 3$

$2x_1 + 3x_2 \leq 4$

$x_1, x_2 \geq 0$.

Then,

(a.) The primal has an optimal solution but the dual does Not have an optimal solution

(b.) Both the primal and the dual have optimal solutions (c.) The dual has an optimal solution but the primal does not have an optimal solution

(d.) Neither the primal nor the dual have optimal solutions

13. Let $f(x) = x^{10} + x - 1, x \in R$ and let $x_k = k,\ k = 0,1,2,....,10$. Then the value of the divided difference $f[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$ is

(a.) –1

(b.) 0

(c.) 1

(d.) 10

14. Let X and Y be jointly distributed random variables having the joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & if \quad x^2 + y^2 \le 1 \\ 0 & otherwise \end{cases}$$

Then $P(Y > \max(X, -X)) =$

(a.) $\frac{1}{2}$

(b.) $\frac{1}{3}$

(c.) $\frac{1}{4}$

(d.) $\frac{1}{6}$

15. Let $X_1, X_2, ...$ be a sequence of independent and identically distributed chi-square random variables, each having 4 degree of freedom. Define $S_n = \sum_{i=1}^{n} X_1^2,\ n = 1,2,....,$

If $\frac{S_n}{n} \xrightarrow{p} \mu$, as $n \to \infty$, then $\mu =$

(a.) 8

(b.) 16

(c.) 24

(d.) 32

16. Let $\{E_n : n = 1,2,....\}$ be a decreasing sequence of Lebesgue measurable sets on R and let F be a Lebesgue measurable set on R such that $E_1 \cap F = \phi$. Suppose that F has Lebesgue measure 2 and the Lebesgue measure of E_n equals $\frac{2n+2}{3n+1}, n = 1, 2,....$ Then the Lebesgue measure of the set $\left(\bigcap_{n=1}^{\infty} E_n\right) \cup F$ equals

(a.) $\frac{5}{3}$

(b.) 2

(c.) $\frac{7}{3}$

(d.) $\frac{8}{3}$

17. The extremum for the variational problem

$$\int_0^{\frac{\pi}{8}}\left((y')^2+2yy'-16y^2\right)dx,\ y(0)=0,\ y\left(\frac{\pi}{8}\right)=1$$ occurs for the curve

(a.) $y=\sin(4x)$

(b.) $y=\sqrt{2}\sin(2x)$

(c.) $y=1-\cos(4x)$

(d.) $y=\frac{1-\cos(8x)}{2}$

18. Suppose $y_p(x)=x\cos(2x)$ is a particular solution of $y''+\alpha y=-4\sin(2x)$.

Then the constant α equals

(a.) –4

(b.) –2

(c.) 2

(d.) 4

19. If $F(s)=\tan^{-1}(s)+k$ is the Laplace transform of some function $f(t), t\geq 0$, then k =

(a.) $-\pi$

(b.) $-\frac{\pi}{2}$

(c.) 0

(d.) $\frac{\pi}{2}$

20. Let $S=\{(0,1,1),(1,0,1),(-1,2,1)\}\subseteq R^3$. Suppose R^3 is endowed with the standard inner product $\langle,\rangle$. Define $M=\{x\in R^3:(x,y)=0 \text{ for all } y\in S\}$. Then the dimension of M equals

(a.) 0

(b.) 1

(c.) 2

(d.) 3

TWO MARKS QUESTIONS (21-75)

21. Let X be an uncountable set and let $\Im = \{U \subseteq X : U = \phi$ or U^c if finite $\}$

Then the topological space $(X, \Im)$

(a.) Is separable

(b.) Is Hausdorff

(c.) Has a countable basis

(d.) Has a countable basis at each point

22. Suppose $(X, \Im)$ is a topological space. Let $\{S_n\}_{n\geq 1}$ be a sequence of subsets of X. Then

(a.) $(S_1 \cup S_2)^\circ = S_1^\circ \cup S_2^\circ$

(b.) $\left(\bigcup_n S_n\right)^\circ = \bigcup_n S_n^\circ$

(c.) $\overline{\bigcup_n S_n} = \bigcup_n \bar{S}_n$

(d.) $\overline{S_1 \cup S_2} = \bar{S}_1 \cup \bar{S}_2$

23. Let (X, d) be a metric space. Consider the metric ρ on X defined by

$\rho(x, y) = \min\{\frac{1}{2}, d(x, y)]\, x, y \in X$.

Suppose $\Im_1$ and $\Im_2$ are topologies on X defined by d and ρ, respectively. Then

(a.) $\Im_1$ is a proper subset of $\Im_2$

(b.) $\Im_2$ is a proper subset of $\Im_1$

(c.) Neither $\Im_1 \subseteq \Im_2$ nor $\Im_2 \subseteq \Im_1$

(d.) $\Im_1 = \Im_2$

24. A basis of $V = \{(x, y, z, w) \in R^4 : x + y - z = 0,\quad y + z + w = 0, 2x + y - 3z - w = 0\}$

(a.) $\{(1,1,-1,0), (0,1,1,1), \{(2,1,-3,1)\}$

(b.) $\{(1,-1,0,1)\}$

(c.) $\{(1,0,1,-1)\}$

(d.) $\{(1,-1,0,1), (1,0,1-1)\}$

25. Consider R^3 with the standard inner product. Let

$S=\{(1,1,1),(2,-1,2),(1,-2,1)\}$.

For a subset W of R^3, let L(W) denote the linear span of W in R^3. Then an orthonormal set T with $L(S)=L(T)$ is

(a.) $\left\{\frac{1}{\sqrt{3}}(1,1,1),\frac{1}{\sqrt{6}}(1,-2,1)\right\}$

(b.) $\{(1,0,0),(0,1,0),(0,0,1)\}$

(c.) $\left\{\frac{1}{\sqrt{3}}(1,1,1),\frac{1}{\sqrt{2}}(1,-1,0)\right\}$

(d.) $\left\{\frac{1}{\sqrt{3}}(1,1,1),\frac{1}{\sqrt{2}}(0,1,-1)\right\}$

26. Let A be a 3×3 matrix. Suppose that the eigen values of A are –1, 0, 1 with respective eigen vectors $(1,-1,0)^t$, $(1,1-2)^t$ and $(1,1,1)^t$. Then 6A equals

(a.) $\begin{bmatrix}-1 & 5 & 2\\ 5 & -1 & 2\\ 2 & 2 & 2\end{bmatrix}$

(b.) $\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$

(c.) $\begin{bmatrix}1 & 5 & 3\\ 5 & 1 & 3\\ 3 & 3 & 3\end{bmatrix}$

(d.) $\begin{bmatrix}-3 & 9 & 0\\ 9 & -3 & 0\\ 0 & 0 & 6\end{bmatrix}$

27. Let $T:R^3\to R^3$ be a linear transformation defined by

$T((x,y,z))=(x+y-z,x+y+z,y-z)$.

Then the matrix of the linear transformation T with respect to the ordered basis $B=\{(0,1,0),(0,0,1),(1,0,0)\}$ of R^3 is 0

(a.) $\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix}$

(b.) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

(c.) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

(d.) $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

28. Let $Y(x) = (y_1(x), y_2(x))$ and let $A = \begin{bmatrix} -3 & 1 \\ k & -1 \end{bmatrix}$.

Further, let S be the set of values of k for which all the solutions of the system of equations $Y'(x) = AY(x)$ tend to zero $x \to \infty$. Then S is given by

(a.) $\{k : k \le -1\}$

(b.) $\{k : k \le 3\}$

(c.) $\{k : k < -1\}$

(d.) $\{k : k < 3\}$

29. Let $u(x, y) = f(xe^y) + g(y^2 \cos(y))$

Where f and g are infinitely differentiable functions. Then the partial differential equation of minimum order satisfied by u is

(a.) $u_{xy} + xu_{xx} = u_x$

(b.) $u_{xy} + xu_{xx} = xu_x$

(c.) $u_{xy} - xu_{xx} = u_x$

(d.) $u_{xy} - xu_{xx} = xu_x$

30. Let C be the boundary of the triangle formed by the points $(1,0,0), (0,1,0), (0,0,1)$.

Then the value of the line integral $\oint_C -2y\,dx + (3x - 4y^2)\,dy + (z^2 + 3y)\,dz$ is

(a.) 0

(b.) 1

(c.) 2

(d.) 4

31. Let X be a complete metric space and let $E \subseteq X$. Consider the following statements:

(S_1) E is compact

(S_2) E is closed and bounded

(S_3) E is closed and totally bounded

(S_4) Every sequence in E has a subsequence converging in E

(a.) S_1

(b.) S_2

(c.) S_3

(d.) S_4

32. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sin(nx)$.

Then the series

(a.) Converges uniformly on R

(b.) Converges point wise but not uniformly on R

(c.) Converges in L1 norm to an integrable function on $[0, 2\pi]$ but does not converge uniformly on R

(d.) Does not converge point wise

33. Let f(z) be an analytic function. Then the value of $\int_0^{2\pi} f(e^{it}) \cos(t)\, dt$ equals

(a.) 0

(b.) $2\pi f(0)$

(c.) $2\pi f'(0)$

(d.) $\pi f'(0)$

34. Let G_1 and G_2 be the images of the disc $\{z \in C : |z+1| < 1\}$ under the transformations $w = \frac{(1-i)z+2}{(1-i)z+2}$ and $w = \frac{(1+i)z+2}{(1+i)z+2}$ respectively. Then

(a.) $G_1 = \{w \in C : \text{Im}(w) < 0\}$ and $G_2 = \{w \in C : \text{Im}(w) ? 0\}$

(b.) $G_1 = \{w \in C : \text{Im}(w) > 0\}$ and $G_2 = \{w \in C : \text{Im}(w) < 0\}$

(c.) $G_1 = \{w \in C : \text{Im}(w) > 2\}$ and $G_2 = \{w \in C : \text{Im}(w) < 2\}$

(d.) $G_1 = \{w \in C : \text{Im}(w) < 2\}$ and $G_2 = \{w \in C : \text{Im}(w) > 2\}$

35. Let $f(z) = 2z^2 - 1$. Then the maximum value of $|f(z)|$ on the unit disc $D = \{z \in C : |z| \le 1\}$ equals

(a.) 1

(b.) 2

(c.) 3

(d.) 4

36. Let $f(z)=\frac{1}{z^2-3z+2}$

Then the coefficient of $\frac{1}{z^3}$ in the Laurent series expansion of $f(z)$ and is

(a.) 0

(b.) 1

(c.) 3

(d.) 5

37. Let $f:C\to C$ be an arbitrary analytic function satisfying $f(0)=0$ and $f(1)=2$. Then

(a.) there exists a sequence $\{z_n\}$ such that $|z_n|$ and $|f(z_n)|>n$

(b.) there exists a sequence $\{z_n\}$ such that $|z_n|$ and $|f(z_n)|<n$

(c.) there exists a bounded sequence $\{z_n\}$ such that $|z_n|$ and $|f(z_n)>n$

(d.) there exists a sequence $\{z_n\}$ such that $z_n \to 0$ and $f(z_n) \to 2$

38. Define $f:C\to C$ by

$$f(z)=\begin{cases}0, & if\ \operatorname{Re}(z)=0\ or\ \operatorname{Im}(z=0,)\\ z, & otherwise\end{cases}$$

Then the set of points where f is analytic is

(a.) $\{z : \operatorname{Re}(z) \neq 0 \text{ and } \operatorname{Im}(z) \neq 0\}$

(b.) $\{z : \operatorname{Re}(z) \neq 0\}$

(c.) $\{z : \operatorname{Re}(z) \neq 0 \text{ or } \operatorname{Im}(z) \neq 0\}$

(d.) $\{z : \operatorname{Im}(z) \neq 0\}$

39. Let U(n) be the set of all positive integers less than *n* and relatively prime to n. Then U(n) is a ground under multiplication modulo n. For n = 248, the number of elements in U(n) is

(a.) 60

(b.) 120

(c.) 180

(d.) 240

40. Let R(*x*) by the polynomial ring in *x* with real coefficients and let I = (x^2 + 1) be the ideal generated by the polynomial x^2 +1 in R[*x*]. Then

(a.) I is a maximal ideal

(b.) I is a prime ideal but NOT a maximal ideal

(c.) I is NOT a prime ideal

(d.) R[*x*] /I has zero divisors

41. Consider Z_5 and Z_{20} as ring modulo 5 and 20, respectively. Then the number of homomorphism φ : Z5 $\to Z_{20}$ is

(a.) 1

(b.) 2

(c.) 4

(d.) 5

42. Let Q be the field of rational number and consider Z_2 as a field modulo 2. Let f(x) = $x_3 - 9x_2 + 9x + 3$.

Then f(x) is

(a.) irreducible over Q but reducible over Z_2

(b.) irreducible over both Q and Z_2

(c.) reducible over Q but irreducible over Z_2

(d.) reducible over both Q and Z_2

43. Consider Z_5 as field modulo 5 and let

$$f(x) = x^5 + 4x^4 + 4x^3 + 4x^2 + x + 1$$

Then the zero of $f(x)$ and over Z_5 are 1 and 3, with respective multiplicity

(a.) 1 and 4

(b.) 2 and 3

(c.) 2 and 2

(d.) 1 and 2

44. Consider the Hilbert space $l^2 = \left\{ x = \{x_n\} : x_n \in R, \sum_{n=1}^{\infty} x_n^2 < \infty \right\}$

Let $E = \left\{ \{x_n\} : |x_n| \le\in \frac{1}{n} \; for\ all\ n \right\}$ be a subset of l^2. Then

(a.) $E^0 = \left\{ x : |x_n| < \frac{1}{n} \; for\ all\ n \right\}$

(b.) $E^0 = E$

(c.) $E^0 = \left\{ x : |x_n| < \frac{1}{n} \; for\ all\ but\ \ finitely\ many\ n \right\}$

(d.) $E^0 = \phi$

45. Let X and Y be normed liner spaces and the T : X $\rightarrow$ Y be a linear map. Then T is continuous if

(a.) Y is finite dimensional

(b.) X is finite dimensional

(c.) T is one to one

(d.) T is onto

46. Let X be a normed linear space and let E_1, $E_2 \subseteq$ X. Define

$$E_1 + E_2 = \{ x + y : x \in E_1, y \in E_2 \}.$$

Then $E_1 + E_2$ is

(a.) open if E_1 or E_2 is open

(b.) NOT open unless both E_1 and E_2 are open

(c.) closed if E_1 or E_2 is closed

(d.) closed if both E_1 and E_2 are closed

47. For each $a \in R$, consider the linear programming problem

Max. $z = x_1 + 2x_2 + 3x_3 + 4x_4$

subject to

$$ax_1 + 2x_2 \le 1$$
$$x_1 + ax_2 + 3x_4 \le 2$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

Let S { $a \in R$: the given LP problem has a basic feasible solution}. Then

(a.) $S = \phi$

(b.) S= R

(c.) S = $(0, \infty)$

(d.) S = $(-\infty, 0)$

48. Consider the linear programming problem

Max. $z = x_1 + 5x_2 + 3x_3$

subject to

$$2x_1 - 3x_2 + 5x_3 \le 3$$
$$3x_1 + 2x_3 \le 5$$
$$x_1, x_2, x_3 \ge 0.$$

Then the dual of this LP problem

(a.) has a feasible solution but does NOT have a basic feasible solution

(b.) has a basic feasible solution

(c.) has infinite number of feasiblel solutions

(d.) has no feasible solution

49. Consider a transportation problem with two warehouses and two markets. The warehouse capacities are $a_1 = 2$ and $a_2 = 4$ and the market demands are $b_1 = 3$ and $b_2 = 3$. let x_{ij} be the quantity shipped from warehouse i to market j and c_{ij} be the corresponding unit cost. Suppose that $c_{11} = 1$, $c_{21} = 1$ and $c_{22} = 2$. Then $(x_{11}, x_{12}, x_{21}, x_{22}) = (2, 0, 1, 3)$ is optimal for every

(a.) $x_{12} \in [1, 2]$

(b.) $x_{12} \in [0, 3]$

(c.) $x_{12} \in [1, 3]$

(d.) $x_{12} \in [2, 4]$

50. The smallest degree of the polynomial that interpolates the data

x	-2	-1	0	1	2	3
f(x)	-58	-21	-12	-13	-6	27

is

(a.) 3

(b.) 4

(c.) 5

(d.) 6

51. Suppose that x_0 is sufficiently close to 3. Which of the following iterations $x_{n+1} = g(x_n)$ will converge to the fixed point $x = 3$?

(a.) $x_{n+1} = -16 + 6x_n + \frac{3}{x_n}$

(b.) $x_{n+1} = \sqrt{3 + 2x_n}$

(c.) $x_{n+1} = \frac{3}{x_n - 2}$

(d.) $x_{n+1} = \frac{x_n^2 - 2}{2}$

52. Consider the quadrature formula, $\int_{-1}^{1} |x| f(x) dx \approx \frac{1}{2}\left[f(x_0) + f(x_1)\right]$

Where x_0 and x_1 are quadrature points. Then the highest degree of the polynomial, for which the above formula is exact, equals

(a.) 1

(b.) 2

(c.) 3

(d.) 4

53. Let A, B and C be three events such that

P(A) = 0.4, P(B) = 0.5, P(A∪B) = 0.6, P(C) = 0.+ and $P(A \cup B \cup C^c) = 0.1$

Then P(A∪B|C)=

(a.) $\frac{1}{2}$

(b.) $\frac{1}{3}$

(c.) $\frac{1}{4}$

(d.) $\frac{1}{5}$

54. Consider two identical boxes B_1 and B_2, where the box B (i = 1, 2) contains i +2 red and 5-i-1 white balls. A fair die is cast. Let the number of dots shown on the top face of the die be N. If N is even or 5, then two balls are down with replacement from the box B_1, otherwise, two balls are drawn with replacement from the box B_2. The probability that the two drawn balls are of different colours is

(a.) $\frac{7}{25}$

(b.) $\frac{9}{25}$

(c.) $\frac{12}{25}$

(d.) $\frac{16}{25}$

55. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variable with

$$P(X_1 = -1) = P(X_1 = 1) = \frac{1}{2}$$

Suppose for the standard normal random variable Z, P(-0.1< Z ≤ 0.1) = 0.08.

If $S_n = \sum_{i=1}^{n^2} X_i$, *then* $\lim_{n\to\infty} P\left(S_n > \frac{n}{10}\right) =$

(a.) 0.42

(b.) 0.46

(c.) 0.50

(d.) 0.54

56. Let $X_1, X_2, \ldots, X_5$ be a random sample of size 5 from a population having standard normal distribution. Let

$$\bar{X} = \frac{1}{5}\sum_{i=1}^{5} X_i, \text{ and } T = \sum_{i=1}^{5}\left(X_i - \bar{X}\right)^2$$

Then $E\left(T^2 \bar{X}^2\right) =$

(a.) 3

(b.) 3.6

(c.) 4.8

(d.) 5.2

57. Let $x_1 = 3.5$, $x_2 = 7.5$ and $x_3 = 5.2$ be observed values of random sample of size three from a population having uniform distribution over the interval (θ, θ +5), where θ ∈ (0, ∞) is unknown and is to be estimated. Then which of the following is NOT a maximum likelihood estimate of θ?

(a.) 2.4

(b.) 2.7

(c.) 3.0

(d.) 3.3

58. The value of $\int_0^\infty \int_{1/y}^\infty x^4 e^{-x^3 y} dxdy$ equals

(a.) $\frac{1}{4}$

(b.) $\frac{1}{3}$

(c.) $\frac{1}{2}$

(d.) 1

59. $\lim_{n\to\infty}\left[(n+1)\int_0^1 x^n \ln(1+x)\,dx\right] =$

(a.) 0

(b.) In 2

(c.) In 3

(d.) ∞

60. Consider the function $f : \mathrm{R} \to \mathrm{R}$ defined by

$$f(x) = \begin{cases} x^4, & \text{if } x \text{ is rational}, \\ 2x^2 - 1, & \text{if } x \text{ is irrational} \end{cases}$$

Let S be the set of points where f is continuous. Then

(a.) S = {1}

(b.) S = {-1}

(c.) S = {-1, 1}

(d.) S = ϕ

61. For a positive real number p, let (f_n: n = 1, 2,....) be a sequence of functions defined on [0, 1] by

$$f_n(x) = \begin{cases} n^{p+1}x, & \text{if } 0 \le x \frac{1}{n} \\ \frac{1}{x^p}, & \text{if } \frac{1}{n} < x \le 1 \end{cases}$$

Let $f(x) = \lim_{n\to\infty} f_n(x), x \in [0,1]$. Then, on [0, 1]

(a.) f is Riemann integrable

(b.) the improper integral $\int_0^1 f(x)\,dx$ converges for p ≥ 1

(c.) the improper integral $\int_0^1 f(x)\,dx$ converges for p < 1

(d.) f_n converges uniformly

62. Which of the following inequality is NOT true for $x \in \left(\frac{1}{4}, \frac{3}{4}\right)$

(a.) $e^{-x} > \sum_{j=0}^{2} \frac{(-x)^j}{j!}$

(b.) $e^{-x} < \sum_{j=0}^{3} \frac{(-x)^j}{j!}$

(c.) $e^{-x} > \sum_{j=0}^{4} \frac{(-x)^j}{j!}$

(d.) $e^{-x} > \sum_{j=0}^{5} \frac{(-x)^j}{j!}$

63. Let u(x, y) be the solution of the Cauchy problem

$xu_x + u_y = 1, u(x, 0) = 2\ln(x), x > 1$

Then $u(e, 1) =$

(a.) -1

(b.) 0

(c.) 1

(d.) e

64. Suppose $y(x) = \lambda \int_0^{2\pi} y(t)\sin(x+t)\,dt, x \in [0, 2\pi]$ has eigenvalue $\lambda = \frac{1}{\pi}$ and $\lambda = -\frac{1}{\pi}$, with corresponding eigenfunctions $y_1(x) = \sin(x) + \cos(x)$ and $y_2(x) = \sin(x) - \cos(x)$, respectively. Then the integral equation

$y(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(x+t)\,dt, x \in [0, 2\pi]$

has a solution when f(x)=

(a.) 1

(b.) cos(x)

(c.) sin(x)

(d.) 1+sin(x) + cos(x)

65. Consider the Neumann problem

$u_{xx} u_{yy} = 0, 0 < x, < \pi, -1 < y < 1$

$u_x(0, y) = u_x = (\pi, y) = 0,$

$u_y(x, -1) = 0, u_y(x, 1) = \alpha + \beta\sin(x)$

The problem admits solution for

(a.) $\alpha = 0, \beta = 1$

(b.) $\alpha = -1, \beta = \frac{\pi}{2}$

(c.) $\alpha = 1, \beta = \frac{\pi}{2}$

(d.) $\alpha = 1, \beta = -\pi$

66. The functional

$\int_0^1 (1+x)(y;)^2 dx, y(0)=0, y(1)=1,$ possesses

(a.) strong maxima

(b.) strong minima

(c.) weak maxima but NOT a strong maxima

(d.) weak minima but NOT a strong minima

67. The value of α for which the integral equation $u(x) = \alpha\int_0^1 e^{x-t} u(t)dt$, has a non-trivial solution is

(a.) -2

(b.) -1

(c.) 1

(d.) 2

68. Let $P_n(x)$ be the Legendre polynomial of degree n and let

$P_{m+1}(0) = -\frac{m}{m+1.} P_m(0), m = 1,2,.....$

If $P_n(0) = -\frac{5}{16}, then \int_{-1}^{1} P_n^2(x)dx =$

(a.) $\frac{2}{13}$

(b.) $\frac{2}{9}$

(c.) $\frac{5}{16}$

(d.) $\frac{2}{5}$

69. For which of the following pair of functions $y_1(x)$ and $y_2(x)$, continuous function $p(x)$ and $q(x)$ can be determined on [-1, 1] such that $y_1(x)$ and $y_2(x)$ give two linearly independent solution of

$y''+p(x)y'+q(x)y = 0, x \in [-1,1]$

(a.) $\frac{2}{13}$

(b.) $\frac{2}{9}$

(c.) $\frac{5}{16}$

(d.) $\frac{2}{5}$

70. Let $J_0(.)$ and $J_1(.)$ be the Bessel functions of the first kind of orders zero and one, respectively.

If $£\left(J_0(t)\right)=\frac{1}{\sqrt{s^2+1}}$, then $£\left(J_1(t)\right)=$

(a.) $\frac{s}{\sqrt{s^2+1}}$

(b.) $\frac{1}{\sqrt{s^2+1}}-1$

(c.) $1-\frac{s}{\sqrt{s^2+1}}$

(d.) $\frac{s}{\sqrt{s^2+1}}-1$

COMMON DATA QUESTIONS

Common Data for Questions 71, 72, 73:

Let P[0, 1] = {p : p is a polynomial function on [0, 1]}. For p∈ P[0, 1] define

$\|P\| = \sup \{|p(x)| : 0 \le x \le 1\}$

Consider the map T:P[0, 1] → P[0, 1] defined by

$$(Tp)(x)=\frac{d}{dx}\left(p(x)\right).$$

71. The linear map T is

(a.) one to one and onto

(b.) one to one but NOT onto

(c.) onto but NOT one to one

(d.) neither one to one nor onto

72. The normal linear space P[0, 1] is

(a.) a finite dimensional normed linear space which is NOT a Banach space

(b.) a finite dimensional Branch space

(c.) an infinite dimensional normed linear space which is NOT a Branch space (d.) an infinite dimensional Banach space

73. The map T is

(a.) closed and continuous

(b.) neither continuous nor closed

(c.) continuous but NOT closed

(d.) closed but NOT continuous

Common Data for Questions 74, 75:

Let X and Y be jointly distributed random variables such that the conditional distribution of Y, given X = x, is uniform on the interval (x – 1, x + 1). Suppose E(X) = 1 and Var (X) = $\frac{5}{3}$.

74. The mean of the random variable Y is

(a.) $\frac{1}{2}$

(b.) 1

(c.) $\frac{3}{2}$

(d.) 2

75. The variance of the random variable Y is

(a.) $\frac{1}{2}$

(b.) $\frac{2}{3}$

(c.) 1

(d.) 2

TWO MARKS QUESTIONS (76-85)

Linked Answer Questions: 76-85 carry two marks each

Statement for Lined Answer Questions 76 and 77:

Suppose the equation $x^2 y'' - xy' + (1+x^2)y = 0$ has a solution of the form $y = x^r \sum_{n=0}^{\infty} c_n x^n, c_0 \neq 0$

76. The indicial equation for r is

(a.) $r^2 - 1 = 0$

(b.) $(r-1)^2 = 0$

(c.) $(r+1)^2 = 0$

(d.) $r^2 + 1 = 0$

77. For $n \geq 2$, the coefficients c_n will satisfy the relation

(a.) $n^2 c_n - c_{n-2} = 0$

(b.) $n^2 c_n + c_{n-2} = 0$

(c.) $c_n - n^2 c_{n-2} = 0$

(d.) $c_n + n^2 c_{n-2} = 0$

Statement for Linked Answer Question 78 and 79:

A particle of mass m slides down without friction along a curve $z = 1 + \frac{x^2}{2}$ in the xz-plane under the action of constant gravity. Suppose the z-axis points vertically upwards. Let $\dot{x}$ and $\ddot{x}$ denote $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$, respectively.

78. The Lagrangian of the motion is

(a.) $\frac{1}{2}m\dot{x}^2(1+x^2) - mg\left(1+\frac{x^2}{2}\right)$

(b.) $\frac{1}{2}m\dot{x}^2(1+x^2) + mg\left(1+\frac{x^2}{2}\right)$

(c.) $\frac{1}{2}mx^2\dot{x}^2 - mg\left(1+\frac{x^2}{2}\right)$

(d.) $\frac{1}{2}m\dot{x}^2(1+x^2) - mg\left(1+\frac{x^2}{2}\right)$

79. The Lagrangian equation of motion is

(a.) $\ddot{x}(1+x^2) = -x(g+\dot{x}^2)$

(b.) $\ddot{x}(1+x^2) = x(g+\dot{x}^2)$

(c.) $\ddot{x} = -gx$

(d.) $\ddot{x}(1-x^2) = x(g+\dot{x}^2)$

Statements for Linked Answer Questions 80 and 81:

Let $u(x, t)$ be the solution of the one dimensional wave equation

$u_{tt} - 4u_{xx} = 0, -\infty < x < \infty, t > 0$

$$u(x,0)=\begin{cases}16-x^2, & |x|\le 4,\\ 0, & otherwise,\end{cases} \quad \text{and}$$

$$u_t(x,0)=\begin{cases}1, & |x|\le 2,\\ 0, & otherwise,\end{cases}$$

80. For 1 < t < 3, u(2, t) =

(a.) $\frac{1}{2}\left[16-(2-2t)^2\right]+\frac{1}{2}\left[1-\min\{1,t-1\}\right]$

(b.) $\frac{1}{2}\left[32-(2-2t)^2-(2+2t)^2\right]+t$

(c.) $\frac{1}{2}\left[32-(2-2t)^2-(2+2t)^2\right]+1$

(d.) $\frac{1}{2}\left[16-(2-2t)^2\right]+\frac{1}{2}\left[1-\max\{1-t,-1\}\right]$

81. The value of $u_t(2, 2)$

(a.) equals -15

(b.) equals -16

(c.) equals 0

(d.) does NOT exist

Statement for Linked Answer Questions 82 and 83:

Suppose $E=\{(x,y):xy\neq 0\}$. Let $f:R^2\to R$ be defined by

$$f(x,y)=\begin{cases}0, & if\ xy=0,\\ y\sin\left(\frac{1}{x}\right)+x\sin\left(\frac{1}{y}\right), & otherwise.\end{cases}$$

Let S_1 be the set of points in R^2 where f_x exists and S_2 be the set of the points in R^2 where f_y exists. Also, let E_1 be the set of points where f_x is continuous and E_2 be the set of points where f_y is continuous.

82. S_1 and S_2 are given by

(a.) $S_1= E\cup\{ x, y):y= 0\}$, $S_2 = E\cup\{(x, y) : x = 0\}$

(b.) $S_1= E\cup\{ x, y):y= 0\}$, $S_2 = E\cup\{(x, y) : y = 0\}$

(c.) $S_1= S_2 = R^2$

(d.) $S_1= S_2 = E\cup\{(0, 0)\}$

83. E_1 and E_2 are given by

(a.) $E_1 = E_2 = S_1 \cap S_2$

(b.) $E_1 = E_2 = S_1 \cap S_2/\{(0,0)\}$

(c.) $E_1 = S_1$, $E_2 = S_2$

(d.) $E_1 = S_2, E_2 = S_1$

Statement for Linked Answer Questions 84 and 85:

Let $A \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{bmatrix}$

and let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigenvalue of A.

84. The triple $(\lambda_1, \lambda_2, \lambda_3)$ equals

(a.) (9, 4, 2)

(b.) (8, 4, 3)

(c.) (9, 3, 3)

(d.) (7, 5, 3)

85. The matrix P such that $P^t AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$ is

(a.) $\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

(b.) $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

(c.) $\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

(d.) $\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$