ENTRANCE EXAMINATION,2005
Ph.D. Mathematics/ Applied Mathematics

TIME: 2 hours **MAX. MARKS: 75**
Part A: 25 Part B: 50

HALL TICKET No. ______________

INSTRUCTIONS

1. Calculators are not allowed.
2. Answer all the 25 questions in Part A. Each correct answer carries **1 mark** and each **wrong** answer carries **minus quarter mark**. Note that this means that wrong answers are penalised by negative marks. So do not gamble.
3. Instructions for answering Part B are given at the beginning of Part B.
4. Do not detach any pages from this answer book. It contains 8 pages. A separate answer book will be provided for Part B.
5. $\mathbb{R}$ always denotes the set of real numbers, $\mathbb{Z}$ the set of integers, $\mathbb{N}$ the set of natural numbers and $\mathbb{Q}$ the set of rational numbers. For any set X, $\mathcal{P}(X)$ is the power set of X.

Part-A

Answer Part A by circling the correct answer. A correct answer gets **1 mark** and a wrong answer gets $-(\mathbf{1/4})$ **mark**.

1. If $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ then the rank of $(A - I)$ (I is the 4×4 identity matrix) is

   a. 4 b. 3 c. 2 d. 1 e. 0

2. The minimal polynomial of $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ is

   a. $(X-2)$ b. $(X-2)^2$ c. $(X-2)^3$ d. $(X-2)^4$ e. $(X-2)^5$

3. Let $S = \{v_1, v_2, ..., v_9\}$ be 9 vectors in $\mathbb{R}^6$. Then
   a. S contains a basis of $\mathbb{R}^6$.
   b. there exist 6 linearly independent vectors in S.
   c. S must span $\mathbb{R}^6$.
   d. there exist 3 linearly independent vectors in S.
   e. none of the above.

4. Let $\sum a_n$ be a convergent series of complex numbers but let $\sum |a_n|$ be divergent. Then it follows that
   a. $a_n \to 0$ but $|a_n|$ does not converge to 0.
   b. the sequence $\{a_n\}$ does not converge to 0.
   c. only finitely many a_n's are 0.
   d. infinitely many a_n's are positive and infinitely many are negative.
   e. none of the above.

5. I. A bounded sequence in $\mathbb{R}$ need not be convergent.
II. A bounded sequence in $\mathbb{R}$ need not have a convergent subsequence.
III. A bounded sequence in $\mathbb{R}$ need not have a constant subsequence.

a. All three statements are true.
b. None of these statements is true.
c. I and II are true but III is false.
d. Only I is true.
e. none of the above.

6. Let $f(x) = \max(\sin x, \cos x)$ for all $x \in \mathbb{R}$. Then
a. f is differentiable on $\mathbb{R}$.
b. f is nowhere differentiable.
c. f is differentiable except at 0.
d. f is differentiable except at a countable set of points.
e. none of the above.

7. Let, for each $x \in [0, 1)$, $x = 0.x_1x_2x_3....$ be the decimal expansion of x not eventually all 9's. Define $f : [0, 1) \to \mathbb{R}$ by $f(x) = x_1$, the first digit in the expansion. Then $\int_0^1 f(x)\, dx =$

a. $4\frac{1}{2}$ b. 10 c. 0 d. 1 e. does not exist.

8. The function $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if}(x, y) \neq (0, 0) \\ 0 & \text{at } (0, 0) \end{cases}$ is
a. continuous at (0,0) but partial derivatives do not exist at (0,0).
b. continuous at (0,0) and partial derivatives exist at (0,0).
c. discontinuous at (0,0) and partial derivatives do not exist at (0,0).
d. discontinuous at (0,0) but partial derivatives exist at (0,0).
e. none of the above.

9. If $\mathbb{R}$ is given the cofinite topology then
   a. $\mathbb{R}$ is compact and connected.
   b. $\mathbb{R}$ is connected but not compact.
   c. $\mathbb{R}$ is compact but not connected.
   d. $\mathbb{R}$ is neither compact nor connected.
   e. $\mathbb{R}$ has a countable base.

10. Let $\mathcal{T} = \{\phi, \mathbb{R}\} \bigcup \{(x, \infty) \mid x \in \mathbb{R}\}$. Then in the topological space $(\mathbb{R}, \mathcal{T})$ the set of integers $\mathbb{Z}$ is
   a. an open set.
   b. a closed set.
   c. a dense set.
   d. an uncountable set.
   e. none of the above.

11. The number of homomorphisms from $C_2 \times C_2 \to C_2$ is (C_n is the cyclic group of order n)

   a. 5 b. 4 c. 3 d. 2 e. 1

12. The number of zero-divisors in the ring of integers modulo 24 is

   a. 20 b. 15 c. 12 d. 8 e. none of the above.

13. If R is a unique factorization domain then
   a. R is a Euclidean domain.
   b. R is a principal ideal domain.
   c. $R[X]$ is a unique factorization domain.
   d. $R[X]$ is a principal ideal domain.
   e. none of the above.

14. The number of proper subfields of F_{32} is

   a. 16 b. 8 c. 4 d. 2 e. 1.

15. An example of a function on $\mathbb{R}$ whose graph does not intersect the x-axis is
a. $f(x) = x^3 - 3x + 2$
b. $f(x) = x^4 + x^2 + 1$
c. $f(x) = x^7 - 2$
d. $f(x) = x^{11} - \frac{x}{2} + 1$
e. none of the above.

16. I. Every Lebesgue measurable function on $\mathbb{R}$ is continuous.
II. Every Lebesgue measurable subset of $\mathbb{R}$ is Borel.
III. The space of continuous functions on $[a, b]$ is dense in $L^3([a, b])$.
a. I and II are true but III is false.
b. I and III are true but II is false.
c. Only II and III are true.
d. Only III is true.
e. None of the above.

17. The indicator function of the irrationals is
a. differentiable everywhere.
b. Riemann integrable.
c. differentiable nowhere.
d. differentiable only at 0.
e. none of the above.

18. For the function $f(z) = \frac{\sin z}{z^2}$ the point $z = 0$ is
a. an essential singularity.
b. a removable singularity.
c. a pole of order 2.
d. a pole of order 1.
e. none of the above.

19. The number of roots of $f(z) = z^5 + 5z^3 + z - 2$ which lie inside the circle of radius $5/2$ centred at the origin is

a. 0 b. 3 c. 5 d. 7 e. none of these.

20. The image of the unit circle under the map $f(z) = 1 + z^2$ is
   a. again the same unit circle.
   b. another circle with a different centre but the same radius.
   c. another circle with the same centre but a different radius.
   d. not a a circle.
   e. none of the above.

21. If A and B are subsets of $\mathbb{R}$, define the distance between them by $d(A, B) = \sup_{n \in \mathbb{R}} |\chi_A(n) - \chi_B(n)|$. ($\chi_A$ is the indicator function of A) Then the metric space $(\mathcal{P}(\mathbb{R}), d)$ is
   a. compact and connected.
   b. compact and normal.
   c. connected and normal.
   d. second countable.
   e. discrete.

22. In the complex Hilbert space $L^2([0, 2\pi])$
   a. the functions $\{e^{inx} \mid n \in \mathbb{Z}\}$ form an orthonormal basis.
   b. the functions $\{\frac{1}{\sqrt{2\pi}} e^{inx} \mid n \in \mathbb{Z}\}$ form an orthonormal basis.
   c. the functions $\{e^{nx} \mid n \in \mathbb{Z}\}$ form an orthonormal basis.
   d. the functions $\{\frac{1}{\sqrt{2\pi}} e^{nx} \mid n \in \mathbb{Z}\}$ form an orthonormal basis.
   e. none of the above.

23. The number of Sylow 2-subgroups in D_7, the dihedral group of order 14, is
   a. 1 b. 2 c. 3 d. 5 e. 7.

24. If y_1 and y_2 are two solutions of $y'' + x^2 y' + (1 - x)y = 0$ on [-1, 1] such that $y_1(0) = 0$, $y_1'(0) = -1$, $y_2(0) = -1$ and $y_2'(0) = 1$ then
   a. y_1, y_2 are linearly independent on [-1,1].
   b. y_1, y_2 are linearly dependent on [-1,1].
   c. y_1, y_2 are linearly dependent on [0,1].
   d. y_1, y_2 are linearly dependent on [-1,0].
   e. none of the above.

25. The ODE $x^2(1-x)^2y'' + (1-x)y' + x^2y = 0$ has
a. both $x = 0$ and $x = 1$ as regular singular points.
b. both $x = 0$ and $x = 1$ as irregular singular points.
c. $x = 0$ as a regular singular point and $x = 1$ as an irregular singular point.
d. $x = 0$ as an irregular singular point and $x = 1$ as a regular singular point.
e. none of the above.

Part - B

There are 15 questions in this part. Each question carries 5 marks. Answer as many as you can. The maximum you can score is 50 marks. Justify your answers. This part must be answered in a separate answer book provided.

1. Let $p : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ be the function defined by $p(A) =$ minimal element of A. Show that
(a) $p(A \cup B) = \min(p(A), p(B))$ and (b) $p(A \cap B) \geq \min(p(A), p(B))$ if $A \cap B \neq \phi$.

2. Show that the function $f(x) = x + \sin x$ defines a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$.

3. What is the characteristic polynomial and minimal polynomial over $\mathbb{Q}$ of the matrix $A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$? Find a vector v such that $\{v, Av, A^2v, A^3v\}$ is a basis of $\mathbb{R}^4$.

4. Let $n \geq 3$ be an odd integer and $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ the non-real nth roots of 1. Show that $(1+\alpha_1^2)(1+\alpha_2^2)...(1+\alpha_{n-1}^2) = 1$.

5. In a commutative ring R with 1, for any subset I of R define $V(I) = \{P \mid P \text{ is a prime ideal containing } I\}$. Show that if I_1 and I_2 are two ideals of R then $V(I_1) \bigcup V(I_2) = V(I_1 \bigcap I_2)$.

6. Show from first principles that a group of order 65 must be cyclic.

7. Define absolute continuity. Give an example of a continuous function that is not absolutely continuous. Show why your example works.

8. For a real number $p > 1$ define the space l^p. Show that the dual space $(l^p)^*$ is isomorphic to l^q where $q = \frac{p}{p-1}$.

9. Determine the Galois group of $\mathbb{Q}(e^{\frac{2\pi i}{7}})$ over $\mathbb{Q}$.

10. Let V be a finite dimensional vector space, $V = V_1 + V_2$, where V_1 and V_2 are two subspaces of V. Let T be a linear transformation on V such that $T(V_1) \subseteq V_2$ and $T(V_2) \subseteq V_1$. Suppose that $T|_{V_1}$ and $T|_{V_2}$ are injective. Show that T is invertible. (Hint: consider T^2).

11. Investigate for solvability the integral equation

$$\phi(x) - \lambda \int_0^1 (2xt - 4x^2)\phi(t)\ dt = 1 - 2x$$

for different values of the parameter λ.

12. Find the extremals with corner point for the functional

$$J[y] = \int_0^2 (y')^2(y' - 1)^2\ dx, \quad y(0) = 0, \quad y(2) = 1.$$

13. Construct the Green's function for the B.V.P. $y'' = -f(x)$, $y(0) = 0$, $y(1) + y'(1) = 2$ and hence write its solution in terms of the Green's function.

14. Consider the non-linear p.d.e. $pq = 1$. Show that two initial strips are possible for the initial curve $x = 2t$, $y = 2t$, $z = 5t$. Find a solution of the equation containing the initial curve.

15. Show that the transformation $Q = p + iaq$, $P = \frac{p-iaq}{2ia}$ is canonical and find a generating function.