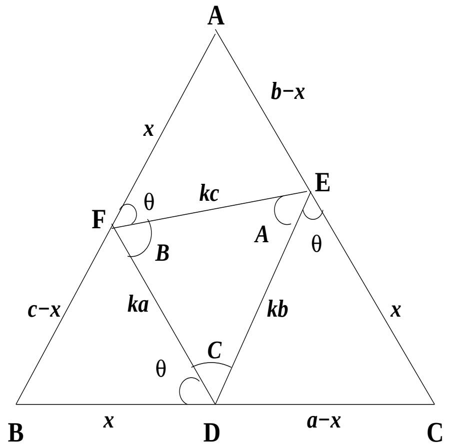
Problems and Solutions, INMO-2011

1. Let D, E, F be points on the sides BC, CA, AB respectively of a triangle ABC such that $BD = CE = AF$ and $\angle BDF = \angle CED = \angle AFE$. Prove that ABC is equilateral.

Solution 1:



Let $BD = CE = AF = x$; $\angle BDF = \angle CED = \angle AFE = \theta$. Note that $\angle AFD = B + \theta$, and hence $\angle DFE = B$. Similarly, $\angle EDF = C$ and $\angle FED = A$. Thus the triangle EFD is similar to ABC. We may take $FD = ka$, $DE = kb$ and $EF = kc$, for some positive real constant k. Applying sine rule to triangle BFD, we obtain

$$\frac{c-x}{\sin\theta} = \frac{ka}{\sin B} = \frac{2Rka}{b},$$

where R is the circum-radius of ABC. Thus we get $2Rk\sin\theta = b(c-x)/a$. Similarly, we obtain $2Rk\sin\theta = c(a-x)/b$ and $2Rk\sin\theta = a(b-x)/c$. We therefore get

$$\frac{b(c-x)}{a} = \frac{c(a-x)}{b} = \frac{a(b-x)}{c}. \tag{1}$$

If some two sides are equal, say, $a = b$, then $a(c-x) = c(a-x)$ giving $a = c$; we get $a = b = c$ and ABC is equilateral. Suppose no two sides of ABC are equal. We may assume a is the least. Since (1) is cyclic in a, b, c, we have to consider two cases: $a < b < c$ and $a < c < b$.

Case 1. $\underline{a < b < c.}$

In this case $a < c$ and hence $b(c-x) < a(b-x)$, from (1). Since $b > a$ and $c - x > b - x$, we get $b(c-x) > a(b-x)$, which is a contradiction.

Case 2. $\underline{a < c < b.}$

We may write (1) in the form

$$\frac{(c-x)}{a/b} = \frac{(a-x)}{b/c} = \frac{(b-x)}{c/a}. \tag{2}$$

Now $a < c$ gives $a - x < c - x$ so that $\dfrac{b}{c} < \dfrac{a}{b}$. This gives $b^2 < ac$. But $b > a$ and $b > c$, so that $b^2 > ac$, which again leads to a contradiction

Thus Case 1 and Case 2 cannot occur. We conclude that $a = b = c$.

Solution 2. We write (1) in the form (2), and start from there. The case of two equal sides is dealt as in Solution 1. We assume no two sides are equal. Using ratio properties in (2), we obtain

$$\frac{a-b}{(ab-c^2)/ca} = \frac{b-c}{(bc-a^2)/ab}.$$

This may be written as $c(a-b)(bc-a^2) = b(b-c)(ab-c^2)$. Further simplification gives $ab^3 + bc^3 + ca^3 = abc(a+b+c)$. This may be further written in the form

$$ab^2(b-c) + bc^2(c-a) + ca^2(a-b) = 0. \tag{3}$$

If $a < b < c$, we write (3) in the form

$$0 = ab^2(b-c) + bc^2(c-b+b-a) + ca^2(a-b) = b(c-b)(c^2-ab) + c(b-a)(bc-a^2).$$

Since $c > b$, $c^2 > ab$, $b > a$ and $bc > a^2$, this is impossible. If $a < c < b$, we write (3), as in previous case, in the form

$$0 = a(b-c)(b^2-ca) + c(c-a)(bc-a^2),$$

which again is impossible.

One can also use inequalities: we can show that $ab^3 + bc^3 + ca^3 \geq abc(a+b+c)$, and equality holds if and only if $a = b = c$. Here are some ways of deriving it:

(i) We can write the inequality in the form

$$\frac{b^2}{c} + \frac{c^2}{a} + \frac{a^2}{b} \geq a + b + c.$$

Adding $a + b + c$ both sides, this takes the form

$$\frac{b^2}{c} + c + \frac{c^2}{a} + a + \frac{a^2}{b} + b \geq 2(a + b + c).$$

But AM-GM inequality gives

$$\frac{b^2}{c} + c \geq 2b, \quad \frac{c^2}{a} + a \geq 2a, \quad \frac{a^2}{b} + b \geq 2a.$$

Hence the inequality follows and equality holds if and only if $a = b = c$.

(ii) Again we write the inequality in the form

$$\frac{b^2}{c} + \frac{c^2}{a} + \frac{a^2}{b} \geq a + b + c.$$

We use b/c with weight b, c/a with weight c and a/b with weight a, and apply weighted AM-HM inequality:

$$b \cdot \frac{b}{c} + c \cdot \frac{c}{a} + a \cdot \frac{a}{b} \geq \frac{(a+b+c)^2}{b \cdot \frac{c}{b} + c \cdot \frac{a}{c} + a \cdot \frac{b}{a}},$$

which reduces to $a + b + c$. Again equality holds if and only if $a = b = c$.

Solution 3. Here is a pure geometric solution given by a student. Consider the triangle BDF, CED and AFE with BD, CE and AF as bases. The sides DF, ED and FE make equal angles θ with the bases of respective triangles. If $B \geq C \geq A$, then it is easy to see that $FD \geq DE \geq EF$. Now using the triangle FDE, we see that $B \geq C \geq A$ gives $DE \geq EF \geq FD$. Combining, you get $FD = DE = EF$ and hence $A = B = C = 60°$.

2. Call a natural number n *faithful*, if there exist natural numbers $a < b < c$ such that a divides b, b divides c and $n = a + b + c$.

(i) Show that all but a finite number of natural numbers are faithful.

(ii) Find the sum of all natural numbers which are **not** faithful.

Solution 1: Suppose $n \in \mathbb{N}$ is faithful. Let $k \in \mathbb{N}$ and consider kn. Since $n = a + b + c$, with $a > b > c$, $c \mid b$ and $b \mid a$, we see that $kn = ka + kb + kc$ which shows that kn is faithful.

Let $p > 5$ be a prime. Then p is odd and $p = (p - 3) + 2 + 1$ shows that p is faithful. If $n \in \mathbb{N}$ contains a prime factor $p > 5$, then the above observation shows that n is faithful. This shows that a number which is not faithful must be of the form $2^\alpha 3^\beta 5^\gamma$. We also observe that $2^4 = 16 = 12 + 3 + 1$, $3^2 = 9 = 6 + 2 + 1$ and $5^2 = 25 = 22 + 2 + 1$, so that 2^4, 3^2 and 5^2 are faithful. Hence $n \in \mathbb{N}$ is also faithful if it contains a factor of the form 2^α where $\alpha \geq 4$; a factor of the form 3^β where $\beta \geq 2$; or a factor of the form 5^γ where $\gamma \geq 2$. Thus the numbers which are not faithful are of the form $2^\alpha 3^\beta 5^\gamma$, where $\alpha \leq 3$, $\beta \leq 1$ and $\gamma \leq 1$. We may enumerate all such numbers:

$$1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120.$$

Among these $120 = 112 + 7 + 1$, $60 = 48 + 8 + 4$, $40 = 36 + 3 + 1$, $30 = 18 + 9 + 3$, $20 = 12 + 6 + 2$, $15 = 12 + 2 + 1$, and $10 = 6 + 3 + 1$. It is easy to check that the other numbers cannot be written in the required form. Hence the only numbers which are not faithful are

$$1, 2, 3, 4, 5, 6, 8, 12, 24.$$

Their sum is 65.

Solution 2: If $n = a + b + c$ with $a < b < c$ is faithful, we see that $a \geq 1$, $b \geq 2$ and $c \geq 4$. Hence $n \geq 7$. Thus $1, 2, 3, 4, 5, 6$ are not faithful. As observed earlier, kn is faithful whenever

n is. We also notice that for odd $n \geq 7$, we can write $n = 1 + 2 + (n - 3)$ so that all odd $n \geq 7$ are faithful. Consider $2n$, $4n$, $8n$, where $n \geq 7$ is odd. By observation, they are all faithful. Let us list a few of them:

$$\begin{aligned} 2n &: \; 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, \ldots \\ 4n &: \; 28, 36, 44, 52, 60, 68, \ldots \\ 8n &: \; 56, 72, \ldots, \end{aligned}$$

We observe that $16 = 12 + 3 + 1$ and hence it is faithful. Thus all multiples of 16 are also faithful. Thus we see that 16, 32, 48, 64, ... are faithful. Any even number which is not a multiple of 16 must be either an odd multiple of 2, or that of 4, or that of 8. Hence, the only numbers not covered by this process are 8, 10, 12, 20, 24, 40. Of these, we see that

$$10 = 1 + 3 + 6, \quad 20 = 2 \times 10, \quad 40 = 4 \times 10,$$

so that 10,20,40 are faithful. Thus the only numbers which are not faithful are

$$1, 2, 3, 4, 5, 6, 8, 12, 24.$$

Their sum is 65.

3. Consider two polynomials $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $Q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ with integer coefficients such that $a_n - b_n$ is a prime, $a_{n-1} = b_{n-1}$ and $a_n b_0 - a_0 b_n \neq 0$. Suppose there exists a rational number r such that $P(r) = Q(r) = 0$. Prove that r is an integer.

Solution: Let $r = u/v$ where $\gcd(u, v) = 1$. Then we get

$$\begin{aligned} a_n u^n + a_{n-1} u^{n-1} v + \cdots + a_1 u v^{n-1} + a_0 v^n &= 0, \\ b_n u^n + b_{n-1} u^{n-1} v + \cdots + b_1 u v^{n-1} + b_0 v^n &= 0. \end{aligned}$$

Subtraction gives

$$(a_n - b_n) u^n + (a_{n-2} - b_{n-2}) u^{n-2} v^2 + \cdots + (a_1 - b_1) u v^{n-1} + (a_0 - b_0) v^n = 0,$$

since $a_{n-1} = b_{n-1}$. This shows that v divides $(a_n - b_n) u^n$ and hence it divides $a_n - b_n$. Since $a_n - b_n$ is a prime, either $v = 1$ or $v = a_n - b_n$. Suppose the latter holds. The relation takes the form

$$u^n + (a_{n-2} - b_{n-2}) u^{n-2} v + \cdots + (a_1 - b_1) u v^{n-2} + (a_0 - b_0) v^{n-1} = 0.$$

(Here we have divided through-out by v.) If $n > 1$, this forces $v \mid u$, which is impossible since $\gcd(v, u) = 1$ ($v > 1$ since it is equal to the prime $a_n - b_n$). If $n = 1$, then we get two equations:

$$\begin{aligned} a_1 u + a_0 v &= 0, \\ b_1 u + b_0 v &= 0. \end{aligned}$$

This forces $a_1 b_0 - a_0 b_1 = 0$ contradicting $a_n b_0 - a_0 b_n \neq 0$. (Note: The condition $a_n b_0 - a_0 b_n \neq 0$ is extraneous. The condition $a_{n-1} = b_{n-1}$ forces that for $n = 1$, we have $a_0 = b_0$. Thus we obtain, after subtraction

$$(a_1 - b_1) u = 0.$$

This implies that $u = 0$ and hence $r = 0$ is an integer.)

4. Suppose five of the nine vertices of a regular nine-sided polygon are arbitrarily chosen. Show that one can select four among these five such that they are the vertices of a trapezium.

Solution 1: Suppose four distinct points P, Q, R, S(in that order on the circle) among these five are such that $\widehat{PQ} = \widehat{RS}$. Then $PQRS$ is an isosceles trapezium, with $PS \parallel QR$. We use this in our argument.

- If four of the five points chosen are adjacent, then we are through as observed earlier. (In this case four points A, B, C, D are such that $\widehat{AB} = \widehat{BC} = \widehat{CD}$.) See Fig 1.

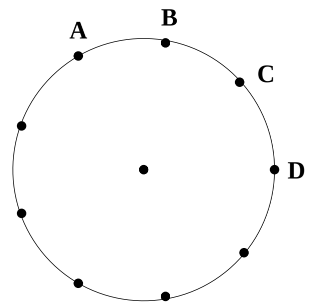


Fig 1.

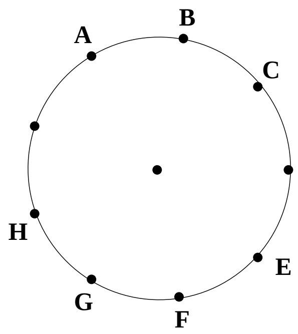


Fig 2.

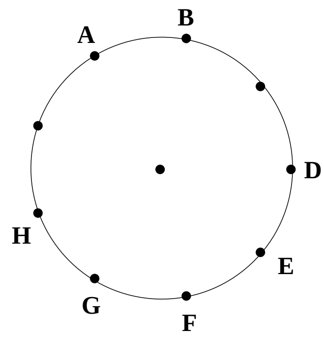


Fig 3.

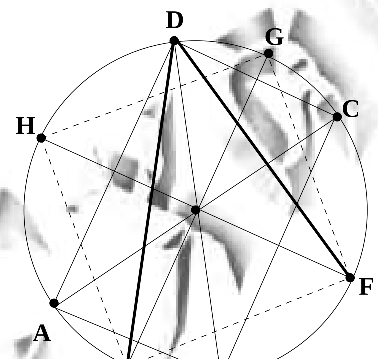
- Suppose only three of the vertices are adjacent, say A, B, C(see Fig 2.) Then the remaining two must be among E, F, G, H. If these two are adjacent vertices, we can pair them with A, B or B, C to get equal arcs. If they are not adjacent, then they must be either E, G or F, H or E, H. In the first two cases, we can pair them with A, C to get equal arcs. In the last case, we observe that $\widehat{HA} = \widehat{CE}$ and $AHEC$ is an isosceles trapezium.
- Suppose only two among the five are adjacent, say A, B. Then the remaining three are among D, E, F, G, H. (See Fig 3.) If any two of these are adjacent, we can combine them with A, B to get equal arcs. If no two among these three vertices are adjacent, then they must be D, F, H. In this case $\widehat{HA} = \widehat{BD}$ and $AHDB$ is an isosceles trapezium.

  Finally, if we choose 5 among the 9 vertices of a regular nine-sided polygon, then some two must be adjacent. Thus any choice of 5 among 9 must fall in to one of the above three possibilities.

Solution 2: Here is another solution used by many students. Suppose you join the vertices of the nine-sided regular polygon. You get $\binom{9}{2} = 36$ line segments. All these fall in to 9 sets of parallel lines. Now using any 5 points, you get $\binom{5}{2} = 10$ line segments. By pigeon-hole principle, two of these must be parallel. But, these parallel lines determine a trapezium.

5. Let $ABCD$ be a quadrilateral inscribed in a circle Γ. Let E, F, G, H be the midpoints of the arcs AB, BC, CD, DA of the circle Γ. Suppose $AC \cdot BD = EG \cdot FH$. Prove that AC, BD, EG, FH are concurrent.

Solution:



Let R be the radius of the circle Γ. Observe that $\angle EDF = \frac{1}{2}\angle D$. Hence $EF = 2R\sin\frac{D}{2}$. Similarly, $HG = 2R\sin\frac{B}{2}$. But $\angle B = 180^\circ - \angle D$. Thus $HG = 2R\cos\frac{D}{2}$. We hence get

$$EF \cdot GH = 4R^2 \sin\frac{D}{2}\cos\frac{D}{2} = 2R^2\sin D = R \cdot AC.$$

Similarly, we obtain $EH \cdot FG = R \cdot BD$.

Therefore

$$R(AC + BD) = EF \cdot GH + EH \cdot FG = EG \cdot FH,$$

by Ptolemy's theorem. By the given hypothesis, this gives $R(AC + BD) = AC \cdot BD$. Thus

$$AC \cdot BD = R(AC + BD) \geq 2R\sqrt{AC \cdot BD},$$

using AM-GM inequality. This implies that $AC \cdot BD \geq 4R^2$. But AC and BD are the chords of Γ, so that $AC \leq 2R$ and $BD \leq 2R$. We obtain $AC \cdot BD \leq 4R^2$. It follows that $AC \cdot BD = 4R^2$, implying that $AC = BD = 2R$. Thus AC and BD are two diameters of Γ. Using $EG \cdot FH = AC \cdot BD$, we conclude that EG and FH are also two diameters of Γ. Hence AC, BD, EG and FH all pass through the centre of Γ.

6. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x+y)f(x-y) = \big(f(x)+f(y)\big)^2 - 4x^2 f(y), \qquad (1)$$

for all $x, y \in \mathbf{R}$, where $\mathbf{R}$ denotes the set of all real numbers.

Solution 1.: Put $x = y = 0$; we get $f(0)^2 = 4f(0)^2$ and hence $f(0) = 0$.
Put $x = y$: we get $4f(x)^2 - 4x^2 f(x) = 0$ for all x. Hence for each x, either $f(x) = 0$ or $f(x) = x^2$.
Suppose $f(x) \not\equiv 0$. Then we can find $x_0 \neq 0$ such that $f(x_0) \neq 0$. Then $f(x_0) = x_0^2 \neq 0$. Assume that there exists some $y_0 \neq 0$ such that $f(y_0) = 0$. Then

$$f(x_0+y_0)f(x_0-y_0) = f(x_0)^2.$$

Now $f(x_0+y_0)f(x_0-y_0) = 0$ or $f(x_0+y_0)f(x_0-y_0) = (x_0+y_0)^2(x_0-y_0)^2$. If $f(x_0+y_0)f(x_0-y_0) = 0$, then $f(x_0) = 0$, a contradiction. Hence it must be the latter so that

$$(x_0^2 - y_0^2)^2 = x_0^4.$$

This reduces to $y_0^2\big(y_0^2 - 2x_0^2\big) = 0$. Since $y_0 \neq 0$, we get $y_0 = \pm\sqrt{2}x_0$.
Suppose $y_0 = \sqrt{2}x_0$. Put $x = \sqrt{2}x_0$ and $y = x_0$ in (1); we get

$$f\big((\sqrt{2}+1)x_0\big)f\big((\sqrt{2}-1)x_0\big) = \big(f(\sqrt{2}x_0) + f(x_0)\big)^2 - 4\big(2x_0^2\big)f(x_0).$$

But $f(\sqrt{2}x_0) = f(y_0) = 0$. Thus we get

$$\begin{aligned} f\big((\sqrt{2}+1)x_0\big)f\big((\sqrt{2}-1)x_0\big) &= f(x_0)^2 - 8x_0^2 f(x_0) \\ &= x_0^4 - 8x_0^4 = -7x_0^4. \end{aligned}$$

Now if LHS is equal to 0, we get $x_0 = 0$, a contradiction. Otherwise LHS is equal to $(\sqrt{2}+1)^2(\sqrt{2}-1)^2x_0^4$ which reduces to x_0^4. We obtain $x_0^4 = -7x_0^4$ and this forces again $x_0 = 0$. Hence there is no $y \neq 0$ such that $f(y) = 0$. We conclude that $f(x) = x^2$ for all x.
Thus there are two solutions: $f(x) = 0$ for all x or $f(x) = x^2$, for all x. It is easy to verify that both these satisfy the functional equation.

Solution 2: As earlier, we get $f(0) = 0$. Putting $x = 0$, we will also get

$$f(y)\big(f(y) - f(-y)\big) = 0.$$

As earlier, we may conclude that either $f(y) = 0$ or $f(y) = f(-y)$ for each $y \in \mathbb{R}$. Replacing y by $-y$, we may also conclude that $f(-y)\big(f(-y) - f(y)\big) = 0$. If $f(y) = 0$ and $f(-y) \neq 0$ for some y, then we must have $f(-y) = f(y) = 0$, a contradiction. Hence either $f(y) = f(-y) = 0$ or $f(y) = f(-y)$ for each y. This forces f is an even function.
Taking $y = 1$ in (1), we get

$$f(x+1)f(x-1) = (f(x)+f(1))^2 - 4x^2 f(1).$$

Replacing y by x and x by 1, you also get

$$f(1+x)f(1-x) = (f(1)+f(x))^2 - 4f(x).$$

Comparing these two using the even nature of f, we get $f(x) = cx^2$, where $c = f(1)$. Putting $x = y = 1$ in (1), you get $4c^2 - 4c = 0$. Hence $c = 0$ or 1. We get $f(x) = 0$ for all x or $f(x) = x^2$ for all x.