Preface

It was a longstanding demand from the students to have a comprehensive reference for all the relevant formulae with regards to Higher Mathematics. We are glad to come out with just that. In this booklet, we have compiled an exhaustive list of formulae from 12 different branches of mathematics and pepped it up with solved examples. This will go a long way to help you in Management Entrance examinations like XAT, IIFT, FMS and JMET etc.

In each chapter, you should brush up the given formulae first and then try to go through the solved example. At the end there is an exercise for your practice that covers all the chapters touched in this booklet.

Please remember, this is not a conventional book on mathematics. So proof of the mentioned formulae is beyond the scope of this book and not relevant for your purpose. This book should be used as a quick, one-stop reference for higher mathematics.

So, what are you waiting for? Start working for few extra marks in Management Entrance examinations like XAT, IIFT, FMS, JMET etc., which asks questions from higher mathematics in quantitative ability section.

Wishing you all the best.
– Career Launcher

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Contents

1 Matrices

A **matrix of order m × n** is defined as the arrangement of mn elements in m rows and n columns.

General representation of matrix A of order m × n is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & --- & a_{mn} \end{pmatrix} = (aij)_{m \times n}, \ 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

Equality

Two matrices A and B are said to be equal, written as A = B, iff

1. they are of the same order, i.e. have the same number of rows and columns, and

2. the corresponding elements of the two matrices are equal.

   If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{o \times p}$, then

   A = B iff m = o, n = p and $a_{ij} = b_{ij} \ \forall \ i, j$

   e.g. if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A \neq B$, but A = C

Algebra of Matrices

Addition: If A and B are two matrices of same type then their **sum** is defined to be the matrix obtained by adding the corresponding elements of A and B. It is denoted by A + B.

If $A = \left(a_{ij}\right)_{m \times n}$ and $B = \left(b_{ij}\right)_{m \times n}$, then $A + B = \left(c_{ij}\right)_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$.

e.g. $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$

1. Matrix addition is commutative. i.e., If A and B are two matrices of same type then A + B = B + A.

2. Matrix addition is associative. i.e., If A, B and C are three matrices of same type then (A + B) + C = A + (B + C).

Subtraction: Same condition as in addition. Resultant matrix is formed by subtracting corresponding elements.

$$A = (a_{ij})_{m\times n}, \ B = (b_{ij})_{m\times n}, \ A - B = (a_{ij} - b_{ij})_{m\times n}$$

e.g. $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Multiplication: Matrices $A = (a_{ij})_{l\times m}, \ B = (b_{ij})_{o\times n}$ can be multiplied iff $m = o$, i.e. numbers of columns in A is equal to the number of rows in B. Resultant in matrix (say C) is of order $(l \times \ n)$ and is given by $AB = C = (c_{ij})_{l\times n}$

where $\mathbf{cij = \displaystyle\sum_{k=1}^{m} a_{ik}b_{kj}}$

e.g. $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2\times1 \ + \ 3\times3 & 2\times2 \ + \ 3\times4 \\ 4\times1 \ + \ 5\times3 & 4\times2 \ + \ 5\times4 \end{pmatrix} = \begin{pmatrix} 11 & 16 \\ 19 & 28 \end{pmatrix}$

Transpose of a Matrix: The matrix obtained by changing the rows of a given matrix A into columns is called **transpose** of A. It is denoted by A^T or A'. If $A = \left(a_{ij}\right)_{m\times n}$ then $A^T = \left(a'_{ij}\right)_{m\times n}$ where $a'_{ji} = a_{ij}$.

e.g. if $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \\ 5 & 0 \end{bmatrix}$.

1. If A is any matrix, then $(A^T)^T = A$.

Conjugate of a Matrix: The conjugate of a given matrix A is obtained by replacing the elements of A by their corresponding complex conjugates. The conjugate of A is denoted by $\overline{A}$. Thus, if $A = \left(a_{ij}\right)_{m\times n}$, then $\overline{A} = \left(\overline{a_{ij}}\right)_{m\times n}$,

e.g. If $A = \begin{bmatrix} 1 & 1+i \\ 3 & 1-i \end{bmatrix}$, then $\overline{A} = \begin{bmatrix} 1 & 1-i \\ 3 & 1+i \end{bmatrix}$.

Tranjugate or Transpose Conjugate of a Matrix: The transpose conjugate of a given matrix is obtained by interchanging the rows and columns of the conjugate matrix of A.

The transpose conjugate of A is denoted by A^*, or by A^θ. Thus, $A^* = \left(\overline{A}\right)'$

e.g. If $A = \begin{bmatrix} 2 & 1+i & 0 \\ 3 & 2 & i \end{bmatrix}$, then $A^* = \begin{bmatrix} 2 & 3 \\ 1-i & 2 \\ 0 & -i \end{bmatrix}$

1.   If A and B are two matrices of same type, then $(A + B)^T = A^T + B^T$.

Adjoint of a Matrix: Let $A = (a_{ij})_{n \times n}$ be a square matrix and C_{ij} the cofactor of a_{ij}. Then the transpose of the matrix B obtained by replacing each element of A by its cofactor in $|A|$ is known as the adjoint of A and is denoted by Adj A.

If, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then Adj $A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

Inverse of a matrix: If A be a n-rowed square matrix, then a square matrix B is said to be an inverse of A if $AB = BA = I_n$. Every matrix has a unique inverse (if exists).

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|A| \neq 0$, then $A^{-1} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$

Different Types of Matrices

1.   A matrix A is said to be a **square matrix** if the number of rows in A is equal to the number of columns in A. An $n \times n$ square matrix is called a **square matrix of type** *n*.

2.   A matrix A is said to be a **rectangular matrix** if the number of rows in A is not equal to the number of columns in A. i.e., A is called a **rectangular matrix** if A is not a square matrix.

3   A matrix A is said to be a **zero matrix** if every element of A is equal to zero. An $m \times n$ zero matrix is denoted by $O_{m \times n}$ or O.

4.   A matrix A is said to be a **row matrix** if A contains only one row.

5.   A matrix, all of whose elements are zero except a_{ii} (i = 1, 2, ...) (which may or may not be zero) in the leading diagonal is called a **diagonal matrix**.

6.   A scalar matrix in which all its diagonal elements are 1 is called **unit matrix**. (Also called **Identity matrix**), we denote n-rowed unit matrix by I_n.

7.   A matrix A is said to be a **column matrix** if A contains only one column.

8.   A diagonal matrix A is said to be a **scalar matrix** if all elements in the principal diagonal are equal.

9. A square matrix A is said to be a **symmetric matrix** if $A^T = A$.

e.g. $A = \begin{bmatrix} 1 & -5 & 7 \\ -5 & 3 & 2 \\ 7 & 2 & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & -5 & 7 \\ -5 & 3 & 2 \\ 7 & 2 & 0 \end{bmatrix} = A$.

Hence, matrix A is symmetric.

10. A square matrix A is said to be a **skew-symmetric**, if $A^T = -A \Rightarrow a_{ij} = -a_{ji} \ \forall \ i, j$.

11. A square matrix A is said to be **orthogonal matrix**, if $AA^T = I = A A^T$,

e.g. the matrix $\dfrac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is an orthogonal matrix, because

$AA^T = \dfrac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \times \dfrac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$\dfrac{1}{9} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$. Similarly, $A^T A = I$.

12. A square matrix A is called **singular** if $|A| = 0$, if $|A| \neq 0$, then it is called **non-singular**.

13. A square matrix A is said to be a **skew matrix** if $A^T = -A$.

Solved Examples

1. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then for every positive integer n, A^n is equal to

a. $\begin{bmatrix} 1+2n & 4n \\ n & 1+2n \end{bmatrix}$ b. $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

c. $\begin{bmatrix} 1-2n & 4n \\ n & 1+2n \end{bmatrix}$ d. None of these

Sol. $A^2 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$

$\Rightarrow A^2 - 2A + I = 0 \Rightarrow A^2 = 2A - I$

Similarly,

$\Rightarrow A^3 = 2A^2 - A = 3A - 2I$

$\Rightarrow A^4 = 3A^2 - 2A = 4A - 3I$

Hence we can say $A^n = nA - (n-1)I$ i.e.

$A^n = \begin{pmatrix} 3n & -4n \\ n & -n \end{pmatrix} - \begin{pmatrix} n-1 & 0 \\ 0 & n-1 \end{pmatrix} = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$

2. $A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 1 & 5 \\ 2 & -5 & 1 \end{bmatrix}$ is a

a. singular matrix   b. non singular matrix

c. symmetric matrix   d. skew symmetric matrix

Sol. $|A| \neq 0$ Hence matrix is non-singular.

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ and $f(x) = 2x^2 - 4x + 5$, then f(A). is equal to

a. $\begin{bmatrix} 19 & -11 \\ -27 & 51 \end{bmatrix}$

b. $\begin{bmatrix} 24 & -16 \\ -32 & 51 \end{bmatrix}$

c. $\begin{bmatrix} 19 & -16 \\ -32 & 51 \end{bmatrix}$

d. None of these

Sol. $f(A) = 2A^2 - 4A + 5I_2$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} \Rightarrow 2A^2 = \begin{bmatrix} 18 & -8 \\ -16 & 34 \end{bmatrix}$$

Also, $-4A = \begin{bmatrix} -4 & -8 \\ -16 & 12 \end{bmatrix}$ and $5I_2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$$\therefore f(A) = 2A^2 - 4A + 5I_2 = \begin{bmatrix} 18 & -8 \\ -16 & 34 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ -16 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 19 & -16 \\ -32 & 51 \end{bmatrix}$$

2 | Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix then the number (ad – bc) is called **determinant** of

A. It is denoted by det A or $|A|$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Determinant of order 3 is given by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32} a_{23}) - a_{12} (a_{21} a_{33} - a_{31} a_{23}) + a_{13} (a_{21} a_{32} - a_{31} a_{22})$$

Properties of Determinants

1. If a_{ij} is an element which is in the ith row and jth column of a, then the determinant of the matrix obtained by deleting the ith row and jth column of A is called **minor** of a_{ij}. It is denoted by M_{ij}.

2. If a_{ij} is an element which is in the ith row and jth column of a square matrix A, then the product of $(-1)^{i+j}$ and the minor of a_{ij} is called **cofactor** of a_{ij}. It is denoted by A_{ij}.

3. The sum of the products of the elements of any row or column of a square matrix A with their corresponding cofactors is called the **determinant** of the matrix A.

4. The sign of the determinant of a square matrix changes if any two rows (or columns) in the matrix are interchanged.

5. If two rows (or columns) of a square matrix are identical, the value of the determinant of the matrix is zero.
    The interchange of any two rows (columns) of a determinant changes only its sign and has no effect on its absolute value.

6. If all the elements of a row (or column) of a square matrix are multiplied by a number k then the value of the determinant of the matrix obtained is k times the determinant of the given matrix.

7. If each element in a row (or column) of a square matrix is the sum of two numbers, then its determinant can be expressed as the sum of the determinants of two square matrices.

Example:

$$\begin{vmatrix} a_1+a_2 & a_3+a_4 & a_5+a_6 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_3 & a_5 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_4 & a_6 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

8. If A and B are two square matrices of same type, then det (AB) = det A. det B.

    The product of two determinants each of order 3 is given by:

    $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

    $$= \begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}$$

    where the rows are multiplied by rows. We can also multiply rows by columns, columns by rows and column by column.

9. If constant multiple of elements of any row(column) be added to (subtracted from) the corresponding elements of some other row(column) of a determinant, then the determinant remains unaltered, i.e. if the operations $R_i \leftarrow R_i + mR_j + nRk. j, k \neq i$ or

    $C_j \leftarrow C_i + mC_j + nC_k . j, k \neq i$

    are performed on the determinant, its value remains unaltered.

Solved Examples

1. If $x \neq y \neq z$ and $\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$, then xyz is equal to

   a. 1   b. –1   c. x + y + z   d. None of these

Sol. $\Delta = 0 \Rightarrow (xyz). \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$

$\Rightarrow (xyz). \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = 0$

$$\Rightarrow (xyz)(y-x)(z-x)\cdot\begin{vmatrix}1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z-x\end{vmatrix}=0$$

$$\Rightarrow (xyz)(y-x)(z-x)=0 \Rightarrow xyz=0$$

2. If a, b and c are all different from zero and $\Delta=\begin{vmatrix}1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c\end{vmatrix}=0$,

then $\left(1+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)=?$

a. abc   b. $\dfrac{1}{abc}$ c. $\dfrac{1}{(a+b+c)}$ d. None of these

Sol. $\Delta = (abc)\cdot\begin{vmatrix}\dfrac{1}{a}+1 & \dfrac{1}{a} & \dfrac{1}{a} \\[2mm] \dfrac{1}{b} & \dfrac{1}{b}+1 & \dfrac{1}{b} \\[2mm] \dfrac{1}{c} & \dfrac{1}{c} & \dfrac{1}{c}+1\end{vmatrix}$

$[R_1 \rightarrow R_1 + R_2 + R_3]$

$= (abc)\times\begin{vmatrix}1+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} & 1+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} & 1+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \\[2mm] \dfrac{1}{b} & \dfrac{1}{b}+1 & \dfrac{1}{b} \\[2mm] \dfrac{1}{c} & \dfrac{1}{c} & \dfrac{1}{c}+1\end{vmatrix}$

$= (abc)\cdot\left(1+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)\cdot\begin{vmatrix}1 & 1 & 1 \\[2mm] \dfrac{1}{b} & \dfrac{1}{b}+1 & \dfrac{1}{b} \\[2mm] \dfrac{1}{c} & \dfrac{1}{c} & \dfrac{1}{c}+1\end{vmatrix} = 0$

$\therefore \Delta=0$ and $abc \neq 0 \Rightarrow \left(1+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)=0$

3

Binomial Theorem, Exponential and Logarithmic Series

Binomial Expansion of $(x + a)^n$, for positive integral index n

$$(x + a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 + ... + {}^nC_r x^{n-r} a^r + ...$$

$$+ {}^nC^{n-1} x^1 a^{n-1} + {}^nC_n x^0 a^n$$

$$= \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$$

where x and a are any real or complex expressions and ${}^nC_0, {}^nC_1..., {}^nC_n$ are called **binomial coefficients.**

This is called the **binomial theorem.**

Properties of **binomial expansion**

1. There are (n + 1) terms in the expansion of $(x + a)^n$, n being a positive integer.

2. In any term of binomial expansion of $(x + a)^n$, the sum of the exponents of x and a is n.

3. The binomial coefficients of terms equidistant from the beginning and the end are equal, i.e. ${}^nC_r = {}^nC_{n-r}$ $(0 \le r \le n)$.

4. The **general term** of the expansion is (r + 1)th term, usually denoted by T_{r+1} and is given by $T_{r+1} = {}^nC_r x^{n-r} a^r$ or , $0 \le r \le 1$.

Important Deductions from Binomial Expansion of $(x + a)^n$

1. $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n = \sum_{r=0}^{n} {}^nC_r x^r$

2. $(1 - x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - ... + (-1)^r {}^nC_r x^r + ... + (-1)^n \cdot {}^nC_n x^n$

3. $\frac{1}{2}\{(1 + x)^n + (1 - x)^n\} = {}^nC_0 + {}^nC_2 x^2 + {}^nC_4 x^4 + ...$

4. $\frac{1}{2}\{(1 + x)^n - (1 - x)^n\} = {}^nC_1 x + {}^nC_3 x^3 + {}^nC_5 x^5 + ...$

5. $(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + ...$

$$+ (-1)^r \frac{n(n+1)(n+2)...(n+r-1)}{r!} x^r + ...$$

6. $(1-x)^{-n} = 1 + nx + \dfrac{n(n+1)}{2!}x^2 + \dfrac{n\ (n+1)(n+2)}{3!}\ x^3$

$\qquad + ... + \dfrac{n(n+1)(n+2)...(n+r-1)}{r!}\ x^r + ...$

7. $(1-x)^n = 1 - nx + \dfrac{n\ (n-1)}{2!}x^2 - \dfrac{n\ (n-1)(n-2)}{3!}\ x^3$

$\qquad + ... + (-1)^r\ \dfrac{n\ (n-1)(n-2)...(n-r+1)}{r!}\ x^r + ...$

Exponential theorem:

If $x \in R$, the series $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ...$ is called exponential series.

Thus $e^x = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$

Important Deductions from the Exponential and Logarithmic Series

1. $e = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + ... = \displaystyle\sum_{n=0}^{\infty} \dfrac{1}{n!}$

2. e is an irrational number and its approximate value is 2.718281828 .....

3. $2 < 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + ... < 3$

4. $e^{-x} = 1 - \dfrac{x}{1!} + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + ... = \displaystyle\sum_{n=0}^{\infty} (-1)^n\ \dfrac{x^n}{n!}$

5. $e = \displaystyle\sum_{n=0}^{\infty} \dfrac{1}{n!} = \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{(n-1)!} = \displaystyle\sum_{n=k}^{\infty} \dfrac{1}{(n-k)!}$.

6. $e^{-1} = \dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} - \dfrac{1}{5!} + ... = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!}$

7. $e^{ax} = 1 + ax + \dfrac{(ax)^2}{2!} + \dfrac{(ax)^3}{3!} + ...$

8. If $x \in R$ and $-1 < x \le 1$ then $\log_e (1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + ...$

9. If $x \in R$ and $|x| < 1$ then $\log_e (1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - ....$

10. $\log 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + ...$

Solved Examples

1. Find the 5th term of $\left(\dfrac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}} - \dfrac{y^{\frac{5}{2}}}{b^{\frac{3}{2}}} \right)^{8}$

Sol. $T_{4+1} = {}^{8}C_{4} \left(\dfrac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}} \right)^{4} \left(\dfrac{-y^{\frac{5}{2}}}{b^{\frac{3}{2}}} \right)^{4}$

$= \dfrac{8!}{4!\ 4!} \cdot \dfrac{x^{6}}{a^{2}} \cdot \dfrac{y^{10}}{b^{6}}$

$= \dfrac{8.7.6.5}{4.3.2} \dfrac{x^{6}y^{10}}{a^{2}b^{6}}$

$= 70 \dfrac{x^{6}y^{10}}{a^{2}b^{6}}$

2. Find the coefficient of x^{r} in $\dfrac{1}{\{(1-x)(3-x)\}}$.

Sol. $\dfrac{1}{\{(1-x)(3-x)\}} = \dfrac{A}{1-x} + \dfrac{B}{3-x} \Rightarrow A = \dfrac{1}{2}, B = -\dfrac{1}{2}$

Therefore, $\dfrac{1}{(1-x)(3-x)} = \dfrac{1}{2}\left[\dfrac{1}{1-x} - \dfrac{1}{3\left(1-\dfrac{x}{3}\right)} \right]$

$= \dfrac{1}{2}(1-x)^{-1} - \dfrac{1}{6}\left(1-\dfrac{x}{3}\right)^{-1}$

Therefore, the coefficient of $x^{r} = \dfrac{1}{2} - \dfrac{1}{6 \times 3^{r}}$

$= \dfrac{1}{2}(1 - 3^{-(r+1)})$

$= \dfrac{1}{2}(1 + x + x^{2} + ... + x^{r} + ...) - \dfrac{1}{6}\left(1 + \dfrac{x}{3} + \left(\dfrac{x}{3}\right)^{r} + ...\right)$

3. Find the coefficient of x^n in the expansion of $\log_e(1+3x+2x^2)$.

Sol. We have: $\log_e (1 + 3x + 2x^2)$

$= \log_e\{(1 + x)(1 + 2x)\} = \log_e(1 + x) + \log_e(1 + 2x)$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{n-1}\frac{x^n}{n} + ... \right) +$$

$$\left(2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - ... + (-1)^{n-1}\frac{(2x)^n}{n} + ... \right)$$

Therefore, coefficient of x^n in $\log_e\left(1 + 3x + 2x^2\right)$

$$\frac{(-1)^{n-1}}{n} + (-1)^{n-1}\frac{2^n}{n} = \frac{(-1)^{n-1}}{n}(1 + 2^n)$$

4 Limits

Definition

The number 'L' is said to be the **limit** of f(x) as x tends to 'a' if **for any** arbitrary $\in\, > 0$, however small, **there exist** a corresponding number $\delta > 0$ such that If(x)–L|$<\in$ whenever $0 < |x - a| < \delta$, i.e. whenever x lies in deleted δ-neighbourhood of a, f(x) lies in $\in$ - neighbourhood of L or equivalently

$$\lim_{x\to a} f(x) = L \text{ iff } \forall \in\, > 0 \ \exists\, \delta > 0 \text{ s.t.}$$

$$L-\in\, < f(x) < L+\in \text{ whenever } x \in (a-\delta, a+\delta) - \{a\}$$

Note that limit is generally written as lim or lt.

Left-Hand Limit (LHL)

A function f(x) tend to a limit L, as x tends to 'a' **from left**, if for any $\in\, > 0$, there exist $\delta > 0$ such that $\left|f(x) - L\right| <\in$ whenever $x \in\,] a - \delta, a [$. In this case, we write

$\underset{x\to a^-}{\text{Lt}}\ f(x) = L$ and say that L is the left-hand limit of f(x).

Right-Hand Limit (RHL)

A function f(x) tend to a limit L, as x tends to 'a' **from right**, if for any $\in\, > 0$, there exist $\delta > 0$ such that $\left|f(x) - L\right| <\in$ whenever $x \in\,] a, a + \delta [$. In this case, we write

$\underset{x\to a^+}{\text{Lt}}\ f(x) = L$ and say that L is the right-hand limit of f(x).

Working rule

$$\text{LHL} = f(a^-) = \lim_{x \to a^-} f(x) = \lim_{0 < h \to 0} f(a - h)$$

$$\text{RHL} = f(a^+) = \lim_{x \to a^+} f(x) = \lim_{0 < h \to 0} f(a + h)$$

Limit exists iff LHL = RHL ,i.e.

$$\lim_{x\to a} f(x) = L \text{ iff } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Rationalisation: Particularly used when numerator or denominator involves radical expressions (square or cube roots etc.)

For example to find $\displaystyle\lim_{a\to 5} \frac{a - 5}{\sqrt{a - 3} - \sqrt{7 - a}}$.

$$= \lim_{a \to 5} \frac{a-5}{\sqrt{a-3} - \sqrt{7-a}} \times \frac{\sqrt{a-3} + \sqrt{7-a}}{\sqrt{a-3} + \sqrt{7-a}}.$$

$$= \lim_{a \to 5} \frac{a-5}{(a-3)-(7-a)} \times (\sqrt{a-3} + \sqrt{7-a}).$$

$$= \lim_{a \to 5} \frac{a-5}{2(a-5)} \times (\sqrt{a-3} + \sqrt{7-a})$$

$$= \lim_{a \to 5} \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Some Standard Limits:

1. $\lim_{x \to a} \{x\} = a$

2. $\lim_{x \to 0} \sin x = 0.$

3. $\lim_{x \to 0} \cos x = 1.$

4. $\lim_{x \to a} \sin x = \sin a, \forall\, a \in R.$

5. $\lim_{x \to a} \cos x = \cos a, \forall\, a \in R.$

6. $\lim_{x \to 0} \tan x = 0.$

7. $\lim_{x \to a} \tan x = \tan a$ for $a \neq (2n+1)\dfrac{\pi}{2}; n \in$ Integer.

8. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1.$

9. $\lim_{x \to 0} \dfrac{\tan x}{x} = 1.$

10. If n is a real number then $\lim_{x \to 0} \dfrac{\sin nx}{x} = n.$

11. If n is a real number then $\lim_{x \to 0} \dfrac{\tan nx}{x} = n.$

12. $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1.$

13. $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1.$

14. $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e.$

15. $\lim_{x \to 0} (1+x)^{1/x} = e.$

Continuity

Working rule:

A function y = f(x) defined on an open interval I is said to be **continuous** at an interior point x = a, if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$, i.e. **LHL = RHL = f(a)** should be in a straight line.

or, $\lim\limits_{x \to a} f(x) = f(a)$, i.e. for f(x) to be continuous at x = a, it must satisfy three conditions:

(1)   f(a) exists, i.e. a lies in the domain of f(x)

(2) $\lim\limits_{x \to a} f(x)$ exists i.e. LHL = RHL and

(3) $\lim\limits_{x \to a} f(x) = f(a)$ the limit equals to the function value.

Continuity of a Function in an Open Interval

A function f(x) is said to be continuous in an open interval (a, b) if f(x) is continuous at every point in (a, b).

Continuity in a Closed Interval

A function f(x) is said to be continuous in a closed interval [a, b], where a < b, if the following three conditions are satisfied.

(a)   f(x) is continuous at left endpoint x = a from right, i.e. $\lim\limits_{x \to a} f(x) = f(a)$

(b)   f(x) is continuous at every interior point of open interval (a, b).

(c)   f(x) is continuous at right endpoint x = b from left, i.e. $\lim\limits_{x \to b-} f(x) = f(b)$.

A function which is not continuous at x = a is said to be discontinuous at x = a and function is said to be discontinuous function.

Solved Examples

1. $\lim\limits_{x \to 0} \left[\dfrac{e^{1/x} - 1}{e^{1/x} + 1} \right] =$

  a. 1 b. 0
  c. -1 d. Does not exist

Sol. Note that $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist as

$$\lim_{x \to 0^+} \frac{1}{x} = \lim_{0 < h \to 0} \frac{1}{0+h} = \infty \text{ and } \lim_{x \to 0^-} \frac{1}{x} = \lim_{0 < h \to 0} \frac{1}{0-h} = -\infty$$

$$\text{RHL} = f(0+) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{1 - e^{-1/(0+h)}}{1 + e^{-1/(0+h)}} = \frac{1-0}{1+0} = 1 \text{ as } e^{-\infty} = 0$$

$$\text{LHL} = f(0-h) = \lim_{h \to 0} f(0-h)$$

$$\lim_{h \to 0} \frac{e^{1/(0-h)} - 1}{e^{1/(0-h)} + 1} = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = \frac{0-1}{0+1} = -1$$

Since LHL $\neq$ RHL. Limit for the function f(x) does not exist.

2. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1} =$

Sol. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1} = \lim\limits_{x \to -\infty} \dfrac{x \sqrt{1 + \dfrac{1}{x^2}}}{x\left(1 + \dfrac{1}{x}\right)}$

$= \lim\limits_{x \to -\infty} \dfrac{\sqrt{1 + \dfrac{1}{x^2}}}{1 + \dfrac{1}{x}} = \dfrac{1}{1+0} = 1$

3. $\lim\limits_{x \to 0} \dfrac{xe^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}} =$

a. 0   b. 1

c. 2   d. Does not exist

Sol. $\text{RHL} = \lim\limits_{x \to 0^+} \dfrac{xe^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}$

$= \lim\limits_{0 < h \to 0} \dfrac{he^{\frac{1}{h}}}{1 + e^{\frac{1}{h}}} = \lim\limits_{0 < h \to 0} \dfrac{h}{1 + e^{\frac{-1}{h}}} = \dfrac{0}{1+0} = 0$

$\text{LHL} = \lim\limits_{x \to 0^-} \dfrac{xe^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}} = \lim\limits_{0 < h \to 0} \dfrac{-he^{\frac{-1}{h}}}{1 + e^{\frac{-1}{h}}} = \dfrac{-0 \times 0}{1+0} = 0 = \text{RHL}$

4. $\lim\limits_{x \to \infty} (4^n + 5^n)^{\frac{1}{n}} =$

Sol. $\lim\limits_{x \to \infty} (4^n + 5^n)^{\frac{1}{n}} = \lim\limits_{x \to \infty} 5\left[1 + \left(\dfrac{4}{5}\right)^n\right]^{\frac{1}{n}} = 5(1+0)^{\circ} = 5$

5. $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\tan 3x}$

Sol. $\lim\limits_{x \to 0} \dfrac{5x\left(\dfrac{\sin 5x}{5x}\right)}{3x\left(\dfrac{\tan 3x}{3x}\right)} = \dfrac{5}{3}\left[\because \lim\limits_{x \to 0}\left(\dfrac{\sin 5x}{5x}\right) = 1 \text{ and } \lim\limits_{x \to 0}\left(\dfrac{\tan 3x}{3x}\right) = 1\right]$

6. Check the continuity of the function $f(x) = \begin{cases} 12 & x \le 4 \\ 4x - [x] & 4 < x < 7 \\ 22 & x \ge 7 \end{cases}$ in

[4, 7].

To check continuity at 4,

RHL $= \lim\limits_{x \to 4^+} 4x - [x] = \lim\limits_{0 < h \to 0} 4(4+h) - [4+h] = 16 - 4 = 12$

LHL $= \lim\limits_{x \to 4^-} f(x) = 12$ and $f(4) = 12$ $\therefore$ LHL $=$ RHL $= f(4)$

To check continuity at 7,

LHL $= \lim\limits_{x \to 7^-} 4x - [x] = \lim\limits_{h \to 0} 4(7-h) - [7-h] = 28 - 6 = 22$ and

RHL $= f(7) = 22 =$ LHL, i.e. $f(x)$ is continuous at $x = 4$ and 7

For checking the continuity in the open interval (4, 7), let us check the continuity at $x = 6$,

RHL $= \lim\limits_{x \to 6^+} 4x - [x] = \lim\limits_{h \to 0} 4(6+h) - [6+h] = 24 - 6 = 18$

LHL $= \lim\limits_{x \to 6^-} 4x - [x] = \lim\limits_{h \to 0} 4(6-h) - [6-h] = 24 - 5 = 19$

$f(6) = 4 \times 6 - [6] = 24 - 6 = 18$

$\because \lim\limits_{x \to 6^+} f(x) \ne \lim\limits_{x \to 6^-} f(x)$, $f(x)$ is not continuous at $x = 6$.

Therefore, function $f(x)$ is not continuous in the closed interval [4, 7].

A function which is not continuous $x = a$ is said to be **discontinuous** at $x = a$ and function is said to be discontinuous function.

5 Differentiation

Differentiability

Derivability at a Point

The function f(x) is said to be derivable or differentiable at x = a, where a is some interior point of domain of f(x), if $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exist,

i.e. $\lim\limits_{0 < h \to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{0 < h \to 0} \dfrac{f(a-h) - f(a)}{-h}$

Derivative of f(x) at x = a is denoted by f'(a) or Df(a) or $\left(\dfrac{d}{dx} f(x) \right)_{x=a}$

Left-Hand Derivative (LHD) of f(x) at x = a, if it exist, is denoted by Lf'(a) and is given by

$\lim\limits_{h \to 0} \dfrac{f(a-h) - f(a)}{-h}$, where h > 0. Lf'(a) is also denoted by f'(a⁻).

Right-Hand Derivative (RHD) of f(x) at x = a, if it exist, is denoted by Rf'(a), and is given by

$\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, where h > 0. Rf'(a) is also denoted by f'(a⁺).

∴ f'(a) exist iff both Lf'(a) and Rf'(a) exist and are equal,
i.e. Lf'(a) = Rf'(a) = f'(a).

If Lf'(a) ≠ Rf'(a) we say that function is not differentiable at x = a.

Derivability in an Interval

A function f(x) is said to be derivable in an open interval (a, b) if f(x) is derivable at every interior point of (a, b).
A function f(x) is said to be derivable in a closed interval [a, b] if

(a) f(x) is derivable at left endpoint x = a from right,

i.e. $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exist.

(b) f(x) is derivable at every point of (a, b).

(c) f(x) is derivable at right endpoint x = b from left,

i.e. $\lim\limits_{h \to 0} \dfrac{f(b-h) - f(b)}{-h}$ exist.

Relation between Continuity and Differentiability

If a function is differentiable at a point, then it is also continuous at that point, but converse is not always true. But if a function is not continuous at a point, then it is not differentiable at that point.

Example of a function which is continuous but not differentiable at a point is, $f(x) = |x|$. This function is continuous at $x = 0$, but it is not differentiable at $x = 0$. Generally, continuous functions which have corners or kinks in their graph are not derivable at those points.

Derived Function

Let $y = f(x)$ be a function with domain D. The derivative of $f(x)$ is denoted

by $f'(x)$ or $\dfrac{dy}{dx}$ or $\dfrac{d}{dx} f(x)$ or $Df(x)$. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, provided the

limit exist. Here h is a small number, either positive or negative. $f'(x)$ is called the **derived function** of $f(x)$.

Algebra of Derivatives

In the following results, k is constant, u and v are functions of x

1. $\dfrac{d}{dx}(k) = 0$

2. $\dfrac{d}{dx}(ku) = k\dfrac{du}{dx}$

3. $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$

4. **Product rule** $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

5. **Quotient rule** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Some Standard Derivatives

1. If c is a constant then $\dfrac{d}{dx}\{c\} = 0$

2. $\dfrac{d}{dx}\{x\} = 1.$

3. $\dfrac{d}{dx}\{x^n\} = nx^{n-1}.$

4. $\dfrac{d}{dx}\{e^x\} = e^x.$

5. $\dfrac{d}{dx}\{a^x\} = a^x \log a.$

6. $\dfrac{d}{dx}\{\log x\} = \dfrac{1}{x}$ for $x > 0.$

7. $\dfrac{d}{dx}\{\sin x\} = \cos x.$

8. $\dfrac{d}{dx}\{\cos x\} = -\sin x.$

9. $\dfrac{d}{dx}\{\tan x\} = \sec^2 x$ for $x \neq (2n+1)\dfrac{\pi}{2}$'$n \in Z.$

10. $\dfrac{d}{dx}\{\cot x\} = -\operatorname{cosec}^2 x$ for $x \neq n\pi, n \in Z.$

11. $\dfrac{d}{dx}\{\sec x\} = \sec x \tan x$ for $x \neq (2n+1)\pi/2, n \in Z.$

12. $\dfrac{d}{dx}\{\operatorname{cosec} x\} = -\operatorname{cosec} x \cot x$ for $x \neq n\pi, \in Z.$

13. If u is a function in x and y is a function in u, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}.$ It is known as **chain rule.**

14. Let f be a differentiable function on [a,b]. Then the derivative f' is a function on [a, b]. If f' is differentiable at x, then the derivative of f' at x is called second derivative of f at x. It is noted by f'' (x) or $f^{(2)}$ (x).
By Induction, n^{th} derivative of f(x) can be defined for all $n \in N$ as

$$f^{(n)}(x) = \dfrac{d}{dx}\left[f^{n-1}(x)\right]$$

15. If y = f(x) then $f^{(n)}$ (x) is denoted by $\dfrac{d^n y}{dx^n}$

16. $\dfrac{d^n y}{dx^n} = \dfrac{d}{dx} \left\{ \dfrac{d^{n-1} y}{dx^{n-1}} \right\}.$

17. If $y = (ax + b)^m$ then $y^n = m(m-1)(m-2)\ldots(m-n+1)(ax+b)^{m-n} \times a^n$

18. If $f(x) = (ax + b)^m$, $m \in Z$, $m > 0$, $n \in N$ then

i) $m < n \Rightarrow f^{(n)}(x) = 0$

ii) $m = n \Rightarrow f^{(n)}(x) = n!\, a^n$

iii) $m > n \Rightarrow f^{(n)} = \dfrac{m!}{(m-n)!}(ax+b)^{m-n}\, a^n.$

19. If $f(x)$ is a polynomial function of degree less than n where $n \in N$ then $f^{(n)}(x) = 0$.

Successive Differentiation

Let $y = f(x0$ be a differentiable function then $y_1 = \dfrac{dy}{dx} = f'(x).$

The derivative of $\dfrac{dy}{dx}$ is called double derivative of $f(x)$ and is denoted

by $\dfrac{d^2 y}{dx^2}$ or y_2 or $f''(x)$. If n is a positive integer, then n^{th} order

derivative of y or $f(x)$ is denoted by y_n or $\dfrac{d^n y}{dx^n}$ or $y^{(n)}$ or $D^n y$ or

$f^{(n)}(x)$.

nth Order Derivative of Some Special Functions

1. $D^n (x)^m = m(m-1)(m-2) \ldots (m-(n-1))x^{m-n} = \dfrac{m!}{(m-n)!} x^{m-n}$

    If $m > n$, when $m = n$, $D^n(x^n) = n!$, whereas when $m < n$, then $D^n\{x^m\} = 0$

2. $D^n (a^{mx}) = a^{mx} (\log a)^n m^n$. Particularly $D^n a^x = a^x(\log a)^n$

3. $D^n e^{ax+b} = a^n e^{ax+b}$

4. If $m \in N$ then $D^n (ax+b)^m = (ax+b)^{m-n} \times a^n \times \dfrac{m!}{(m-n)!}$ if $n < m$.

    If $n > m$, then $D^n(ax+b)^m = 0$, whereas if $n = m$, then

    $D^n (ax+b)^m = n!a^n$

5. $D^n (ax+b)^{-1} = (-1)^n \ n! \ a^n (ax+b)^{-n-1}$ or

    $D^n \left(\dfrac{1}{ax+b} \right) = \dfrac{(-1)^n n!}{(ax+b)^{n+1}} a^n \Rightarrow D^n \left(\dfrac{1}{x} \right) = \dfrac{(-1)^n n!}{(x)^{n+1}}$

6. $D^n \log_e (ax+b) = \dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

7. $D^n \sin(ax+b) = a^n \sin\left(ax+b+\dfrac{n\pi}{2} \right)$

8. $D^n \cos(ax+b) = a^n \cos\left(ax+b+\dfrac{n\pi}{2} \right)$

9. $D^n \left\{ e^{ax} \sin(bx+c) \right\} = \left(a^2+b^2\right)^{n/2} \times e^{ax} \sin\left\{ bx+c+n \tan^{-1}\dfrac{b}{a} \right\}$

10. $D^n \left\{ e^{ax} \cos(bx+c) \right\} = \left(a^2+b^2\right)^{n/2} \times e^{ax} \cos\left\{ bx+c+n \tan^{-1}\dfrac{b}{a} \right\}$

Solved Examples

1.   If $y = \dfrac{1}{x^x}$, find $\dfrac{d^2 y}{dx^2}$

Sol. $y = x^{-x}$

$\Rightarrow \log y = -x \log x$

$\Rightarrow \dfrac{1}{y} \dfrac{dy}{dx} = -x \cdot \dfrac{1}{x} + \log x \, (-1)$

$\Rightarrow \dfrac{dy}{dx} = (-\log x - 1) \, y$

$\Rightarrow \dfrac{d^2 y}{dx^2} = -\dfrac{dy}{dx}(\log x + 1) - \dfrac{y}{x} = -(-\log x - 1) \cdot \dfrac{1}{x^x}(\log x + 1) - \dfrac{1}{x^x} \cdot \dfrac{1}{x}$

$= (1 + \log x)^2 \cdot x^{-x} - x^{-(x+1)}$

2.   For $y = \log \{ \, 1 - \log(\, 1 - x \,) \, \}$, the value of $\dfrac{d^3 y}{dx^3}$ at $x = 0$ is

   a. 1   b. 2   c. $-\dfrac{1}{6}$ d. -1

Sol. $1 - e^y = \log(1 - x)$ at $x = 0$, $e^y = 1$

$\Rightarrow e^y y' = \dfrac{1}{1 - x}$ at $x = 0$, $y' = 1$

$\Rightarrow e^y \left(y' \right)^2 + e^y y'' = \dfrac{1}{(1 - x)^2}$ at $x = 0$, $e^y = 1$, $y' = 1$, $y'' = 0$

$\Rightarrow e^y \left(y' \right)^3 + 2e^y y' y'' + e^y y' y'' + e^y y''' = \dfrac{2}{(1 - x)^3}$

At $x = 0$, $e^y = 1$, $y' = 1$, $y'' = 0$, $y''' = 1$

3. $\dfrac{d}{dx}\left(\dfrac{e^{2x}}{\log x}\right)=$

Sol. $\dfrac{d}{dx}\left(\dfrac{e^{2x}}{\log x}\right)=\dfrac{\log x\,\dfrac{d}{dx}\left(e^{2x}\right)-e^{2x}\dfrac{d}{dx}\left(\log x\right)}{\left(\log x\right)^2}=\dfrac{2\log x\cdot e^{2x}-\dfrac{e^{2x}}{x}}{\left(\log x\right)^2}$

$$=\dfrac{\left(2x\log x-1\right)e^{2x}}{x\left(\log x\right)^2}$$

4. $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x-1...\infty}}}}$

Sol. $\Rightarrow\dfrac{dy}{dx}+\dfrac{1}{y^2}\dfrac{dy}{dx}=1\ \Rightarrow\dfrac{dy}{dx}\left(1+\dfrac{1}{y^2}\right)=1\Rightarrow\dfrac{dy}{dx}=\dfrac{y^2}{y^2+1}$

5. $\sin y=x\sin(a+y)$, then $\dfrac{dy}{dx}$ is equal to

Sol. $x=\dfrac{\sin y}{\sin(a+y)}$

Differentiate w.r.t. y'

$\dfrac{dx}{dy}=\dfrac{\sin(a+y)\cos y-\cos(a+y)\sin y}{\sin^2(a+y)}=\dfrac{\sin(a+y-y)}{\sin^2(a+y)}=\dfrac{\sin a}{\sin^2(a+y)}$

$\Rightarrow\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$

6. $y=\tan^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$, then $\dfrac{dy}{dx}$ is

    a. independent of x   b. $\tan^{-1}\left(\dfrac{\sin x+1}{\sin x-1}\right)$

    c. $\dfrac{x}{2}$ d. $\dfrac{\pi}{2}+x$

Sol. $\dfrac{dy}{dx}$ is independent of x, as

$y=\tan^{-1}\left(\dfrac{1+\cos x}{\sin x}\right)=\tan^{-1}\left(\cot\dfrac{x}{2}\right)=\dfrac{\pi}{2}-\dfrac{x}{2}$

6 Integration

Indefinite Integrals:

If $\dfrac{d}{dx}(g(x)) = f(x)$, then $\int f(x)\,dx = g(x) + c$, where c is any arbitrary real number and is called constant of integration. The function f(x) in $\int f(x)\,dx$ is called the **integrand**, the function g(x) is called **integral** or **anti-derivative** or **primitive** of the function f(x) w. r. t. x. If F(x) is an anti-derivative of f(x) then F(x) + c, c $\in$ R is called **indefinite integral** of f(X) with respect to x. It is denoted by $\int f(x)\,dx$. The real number c is called **constant of integration**.

Some Standard Indefinite Integrals

1.   If $n \neq -1$ then $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c.$

2. $\int dx = x + c.$

3. $\int \dfrac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + c.$

4. $\int \dfrac{1}{x}\,dx = \log|x| + c.$

5. $\int e^x\,dx = e^x + c.$

6.   If $a > 0$, and $a \neq 1$, then $\int a^x\,dx = \dfrac{a^x}{\log a} + c.$

7. $\int \cos x\,dx = \sin x + c.$

8. $\int \sin x\,dx = -\cos x + c.$

9. $\int \sec^2 x\,dx = \tan x + c.$

10. $\int \cosec^2 x \, dx = -\cot x + c.$

11. $\int \sec x \tan x \, dx = \sec x + c.$

12. $\int \cosec x \cot x \, dx = -\cosec x + c.$

13. $\int \dfrac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c = -\cos^{-1} x + c$

14. $\int \dfrac{1}{1+x^2} \, dx = \tan^{-1} x + c = -\cot^{-1} x + c.$

15. If $x > 1$, then $\int \dfrac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + c = -\cosec^{-1} x + c$ and if

$x < -1$ then $\int \dfrac{1}{x\sqrt{x^2-1}} \, dx = -\sec^{-1} x + c = \cosec^{-1} x + c.$

16. $\int \tan x \, dx = \log|\sec x| + c.$

17. $\int \cot x \, dx = \log|\sin x| + c.$

18. $\int \dfrac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\dfrac{x}{a}\right) + c.$

19. $\cdot \int \dfrac{1}{\sqrt{a^2+x^2}} \, dx = \ln\left(x + \sqrt{a^2+x^2}\right) + c = \sinh^{-1}\dfrac{x}{a} + c.$

Substitution: $x = a \tan\theta$ or $a \sinh\theta$

20. $\int \dfrac{1}{\sqrt{x^2-a^2}} \, dx = \ln\left|x + \sqrt{x^2-a^2}\right| + c = \cosh^{-1}\dfrac{x}{a} + c.$

Substitution: $x = a \sec\theta$ or $a \cosh\theta$

21. $\int \dfrac{1}{a^2-x^2} \, dx = \dfrac{1}{a} \tan^{-1}\left(\dfrac{x}{a}\right) + c.$

22. $\displaystyle\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \log\left(\frac{a + x}{a - x}\right) + c.$

23. $\displaystyle\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln\left(x + \sqrt{a^2 + x^2}\right) + c\,.$

Substitution: x = a tanθ or a cotθ or a sinhθ.

24. $\displaystyle\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + c\,.$

Substitution: x = a secθ or a coshθ.

25. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\,.$

Substitution: x = a tan θ.

26. Integration By Parts : If f(x) and g(x) are two integrable functions then

$$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int f'(x)\left[\int g(x)dx\right]dx.$$

27. $\displaystyle\int e^x \left[f(x) + f'(x)\right] dx = e^x\, f(x) + c$

Integrals of Various Type

Type 1

$$\int \frac{dx}{ax^2 + bx + c}\,,\ \int \frac{dx}{\sqrt{ax^2 + bx + c}}\,,\ \int \sqrt{ax^2 + bx + c}\ dx$$

Express $ax^2 + bx + c$ in the form of perfect square and then apply the standard results.

Type 2

$$\int \frac{px + q}{ax^2 + bx + c}dx,\ \int \frac{px + q}{\sqrt{ax^2 + bx + c}}dx,$$

$$\int (px + q)\sqrt{ax^2 + bx + c}\ dx\,,\ \int \frac{p(x)}{ax^2 + bx + c}dx\,,\ \text{where p(x) is any polynomial of}$$

x

In first three of these, let $px + q = \lambda \dfrac{d}{dx}\left(ax^2 + bx + c\right) + \mu$ where λ and μ are constants, which you can find by comparing the coefficients, this will break the given integral into two integrals, where in first integral substitute $ax^2 + bx + c = y$ and second is of type 1. Reduce fourth integral to first type, by dividing $p(x)$ by $ax^2 + bx + c$.

Type 3

$$\int \frac{dx}{a + b\sin^2 x} \,,\, \int \frac{dx}{a + b\cos^2 x} \,,\, \int \frac{dx}{a\sin^2 x + b\cos^2 x} \,,$$

$$\int \frac{dx}{(a\sin x + b\cos x)^2} \,,$$

$$\int \frac{f(\tan x)dx}{a\sin^2 x + b\sin x \cos x + c\cos^2 x + d} \quad \text{where } f(\tan x) \text{ is polynomial in } \tan x$$

Divide numerator and denominator by $\cos^2 x$ and then substitute $\tan x = t$, $\sec^2 x \, dx = dt$, then integral reduces to Type 2.

Type 4

$$\int \frac{dx}{a + b\sin x} \,,\, \int \frac{dx}{a + b\cos x} \,,\, \int \frac{dx}{a\sin x + b\cos x + c} \,,$$

$$\int \frac{c\sin x + d\cos x}{a\sin x + b\cos x} \, dx \,,\, \int \frac{d\sin x + e\cos x + f}{a\sin x + b\cos x + c} \, dx$$

In first three of these substitute

$$\tan \frac{x}{2} = t \,,\, dx = \frac{2dt}{1 + t^2} \,,\, \cos x = \frac{1 - t^2}{1 + t^2} \,,\, \sin x = \frac{2t}{1 + t^2}$$

In fourth, express numerator as

$$\lambda\left(\text{deno min ator}\right) + \mu\left(\frac{d}{dx}(\text{deno min ator})\right) \text{by comparing the coefficients}$$

find the value of constants. Then integral value is $\lambda x + \mu \ln |\text{denominator}| + c$. Similarly in fifth, express numerator as

$$\lambda\left(\text{deno min ator}\right) + \mu\left(\frac{d}{dx}(\text{deno min ator})\right) + v$$

Definite Integrals

Let $f(x)$ be a function defined on [a, b]. If $\int f(x)\,dx = F(x),$ then

$F(b) - F(a)$ is called the **definite integral** of $f(x)$ over [a, b]. It is denoted by

$\int\limits_{a}^{b} f(x)\,dx.$ The real number a is called the **lower limit** and the real number b is called the **upper limit**.

Some Standard Definite Integrals

1. $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

2. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

3. If $a < c < b$ then $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx. + \int_c^b f(x)\,dx.$$

4. $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

5. $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

6. $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a (x)\,dx, & \text{if } f(x) \text{ is an even function.} \\ 0, & \text{if } f(x) \text{ is an odd function.} \end{cases}$$

7. $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a (x)\,dx, & \text{if } f(2a-x) = f(x). \\ 0, & \text{if } f(2a-x) = -f(x). \end{cases}$$

8. $$\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx \text{ , if } f(x) \text{ is periodic with period } T \text{ and } n \in Z$$

    Hint: Substitute $nT + y$ for x.

9. If $f(x)$ is a periodic function with period T and $a \in R^+$, $m, n \in Z$, then

    (a) $$\int_T^{a+T} f(x)\,dx = \int_0^a f(x)\,dx \text{ , by substituting } T + y \text{ for x}$$

    (b) $$\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$$

    (c) $$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx \text{ i.e. } \int_a^{a+T} f(x)\,dx \text{ is independent of a.}$$

Solved Examples

1. $\displaystyle\int 4^x e^x dx$

Sol. $\displaystyle\int (4e)^x dx = \frac{(4e)^x}{\ln(4e)} + c$

2. $\displaystyle\int \frac{2x^4 + 3}{x^2 + 1} dx$

Sol. $\displaystyle\int \frac{2x^4 + 2x^2 - 2x^2 - 2 + 5}{x^2 + 1}$

$\displaystyle= \int \frac{2x^4 + 2x^2}{x^2 + 1} dx + \int \frac{-2x^2 - 2}{x^2 + 1} dx + \int \frac{5}{x^2 + 1} dx$

$\displaystyle= \int 2x^2 dx - \int 2 dx + \int \frac{5}{x^2 + 1} dx$

$\displaystyle= \frac{2x^3}{3} - 2x + 5 \tan^{-1}(x) + c$

3. $\displaystyle\int \frac{dx}{3\sin x + 4\cos x}$

Sol. $3\sin x + 4\cos x = 5(\sin x \cos\alpha + \cos x \sin\alpha) = 5\sin(x + \alpha)$ where

$\alpha = \tan^{-1}\left(\dfrac{4}{3}\right)$

$\displaystyle= \frac{1}{5}\int \frac{dx}{\sin(x+\alpha)} = \frac{1}{5}\int \cos ec\,(x+\alpha)dx == \frac{1}{5}\ell n\left|\tan\frac{(x+\alpha)}{2}\right| + c$

4. $\displaystyle\int \frac{2x - 3}{3x^2 + 4x + 5} dx$

Sol. Put $2x - 3 = \lambda\left(\dfrac{d}{dx}(3x^2 + 4x + 5)\right) + \mu = \lambda(6x + 4) + \mu$, given integral

becomes $\displaystyle\lambda\int \frac{6x + 4}{3x^2 + 4x + 5} + \mu\int \frac{dx}{3x^2 + 4x + 5}$

Integrating $\displaystyle\int \frac{dx}{3x^2 + 4x + 5}$

$\displaystyle= \frac{1}{3}\int \frac{dx}{x^2 + \frac{4}{3}x + \frac{5}{3}} = \frac{1}{3}\int \frac{dx}{\left(x + \frac{2}{3}\right)^2 + \left(\frac{\sqrt{11}}{3}\right)^2} = \frac{1}{3}\cdot\frac{3}{\sqrt{11}}\tan^{-1}\left(\frac{3x + 2}{\sqrt{11}}\right)$

Comparing the coefficients, we get $\lambda = \dfrac{1}{3}\,\mu, = -\dfrac{13}{3}$

$$= \dfrac{1}{3}\,\text{In}\,|\,3x^2 + 4x + 5\,| - \dfrac{13}{3\sqrt{11}}\,\tan^{-1}\left(\dfrac{3x+2}{\sqrt{11}}\right) + C$$

5. $\displaystyle \int \dfrac{1}{(x-1)\sqrt{x^2+4}}\,dx$

Sol. Let $x - 1 = \dfrac{1}{t} \Rightarrow x = \dfrac{1}{t} + 1\,\,dx = \dfrac{-1}{t^2}\,dt$

$$I = \int \dfrac{-\dfrac{1}{t^2}\,dt}{\dfrac{1}{t}\sqrt{\left(\dfrac{1}{t}+1\right)^2 + 4}} = -\int \dfrac{dt}{\sqrt{(t+1)^2 + 4t^2}} = -\int \dfrac{dt}{\sqrt{5t^2 + 2t + 1}}$$

$$=$$

$$\int \dfrac{-\dfrac{1}{t^2}\,dt}{\dfrac{1}{t}\sqrt{\left(\dfrac{1}{t}+1\right)^2 + 4}} = -\int \dfrac{dt}{\sqrt{(t+1)^2 + 4t^2}} = \dfrac{-1}{\sqrt{5}}\int \dfrac{1}{\sqrt{\left(t+\dfrac{1}{5}\right)^2 + \left(\dfrac{2}{5}\right)^2}}\,dt$$

$$= \dfrac{-1}{\sqrt{5}}\,\text{In}\left|\left(t+\dfrac{1}{5}\right) + \sqrt{t^2 + \dfrac{2}{5}t + \dfrac{1}{5}}\right| + C$$

$$= \dfrac{-1}{\sqrt{5}}\,\text{In}\left|\dfrac{1}{x-1} + \dfrac{1}{5} + \sqrt{\dfrac{1}{(x-1)^2} + \dfrac{2}{5(x-1)} + \dfrac{1}{5}}\right| + C$$

$$= \dfrac{-1}{\sqrt{5}}\,\text{In}\left|\dfrac{1}{x-1} + \dfrac{1}{5} + \sqrt{\dfrac{x^2+4}{5(x-1)^2}}\right| + C$$

6. $\displaystyle \int \dfrac{dx}{\sqrt{7 + 4x - 2x^2}}$

Sol. $\dfrac{1}{\sqrt{2}}\displaystyle\int \dfrac{dx}{\sqrt{\dfrac{7}{2} - (x^2 - 2x)}} = \dfrac{1}{\sqrt{2}}\int \dfrac{dx}{\sqrt{\dfrac{9}{2} - (x-1)^2}}$

$$= \dfrac{1}{\sqrt{2}}\int \dfrac{dx}{\sqrt{\left(\dfrac{3}{\sqrt{2}}\right)^2 - (x-1)^2}}$$

$$= \dfrac{1}{\sqrt{2}}\,\sin^{-1}\left(\dfrac{\sqrt{2}(x-1)}{3}\right) + C$$

7. $\displaystyle\int_{0}^{\pi/2} \frac{dx}{1+\sqrt{\cot x}}$

Sol. $\displaystyle\int_{0}^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ ... (i)

$$\int_{0}^{\pi/2} \frac{\sqrt{\sin\left(\dfrac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\dfrac{\pi}{2}-x\right)}+\sqrt{\cos\left(\dfrac{\pi}{2}-x\right)}}\,dx \left[\because \int_{0}^{a}f(x)dx = \int_{0}^{a}f(a-x)dx\right]$$

$$= \int_{0}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \qquad\qquad ... (ii)$$

Adding (i) and (ii), we get

$$2I = \int_{0}^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \text{ i.e.} 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

8. $\displaystyle\int_{1}^{2} \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx$

Sol. $\displaystyle\int_{1}^{2} \frac{\sqrt{3-x}}{\sqrt{3-(3-x)}+\sqrt{3-x}}\,dx = \int_{1}^{2} \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\,dx$

$$2I = \int_{1}^{2} \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx + \int_{1}^{2} \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\,dx = \int_{1}^{2} \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{3-x}+\sqrt{x}}\,dx$$

$$2I = \int_{1}^{2} dx \quad \Rightarrow I = \frac{2-1}{2} = \frac{1}{2}$$

9. $\displaystyle\int_{-3}^{3} |x|\,dx$

Sol. $\displaystyle\int_{-3}^{3} |x|\,dx = \int_{-3}^{0} |x|\,dx + \int_{0}^{3} |x|\,dx$

If $x < 0$, $|x| = -x$

$x > 0, |x| = -x$

$$= \int_{-3}^{0} -x \, dx + \int_{0}^{3} x \, dx = -\frac{x^2}{2} \Bigg]_{-3}^{0} + \frac{x^2}{2} \Bigg|_{0}^{3}$$

$$= 0 - \left(-\frac{9}{2} \right) + \frac{9}{2}$$

$$= 9$$

Same can be obtained as follows

$|x|$ is an even function.

$$\int_{-3}^{3} |x| \, dx = 2\int_{0}^{3} |x| \, dx = 2\int_{0}^{3} x \, dx = 2 \left[\frac{x^2}{2} \right]_{0}^{3} = \frac{2 \times 9}{2} = 9$$

10. $$\int_{0}^{\pi/2} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$$

Sol.   Dividing the numerator and the denominator by $\cos^2 x$,

$$\int_{0}^{\pi/2} \frac{\sec^2 x \, dx}{a^2 \tan^2 x + b^2}$$

Let $\tan x = t$, $\sec^2 x \, dx = dt$, $x \to 0 \Rightarrow t \to 0$, $x \to \dfrac{\pi}{2} \Rightarrow t \to \infty$

$$\int_{0}^{\infty} \frac{dt}{a^2 t^2 + b^2} = \frac{1}{a^2} \int_{0}^{\infty} \frac{dt}{t^2 + \dfrac{b^2}{a^2}}$$

$$= \frac{1}{a^2} \cdot \left| \frac{a}{b} \tan^{-1} \left(\frac{at}{b} \right) \right]_{0}^{\infty} = \frac{1}{ab} \left[\tan^{-1} \frac{at}{b} \right]_{0}^{\infty} = \frac{1}{ab} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{2ab}$$
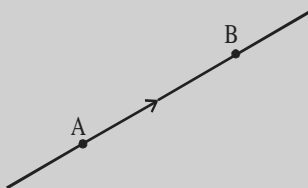
7 Vectors

Scalar and Vector Quantities

(i) **Scalar:** A quantity which has only magnitude.
Examples: Length, mass, volume, time, temperature etc., are all scalars.

(ii) **Vector:** A quantity which has magnitude as well as direction or a directed line segment is called a vector.
**Examples:** Displacement, velocity, acceleration, force etc., are all vectors.

Representation of Vectors

Let A be an arbitary point in space and B any other point. Then the line segment AB has both magnitude and direction, i.e. from A to B. Point A is called the initial point (or tail) and B is called the terminating point (or head) of vector $\overrightarrow{AB}$. A vector is denoted by $\vec{a}$ and its magnitude by $\left|\vec{a}\right|$ or a.

Properties of Vectors

1. A vector of unit magnitude is called **unit vector**. A unit vector in the

direction of $\vec{a}$ is given by $\dfrac{\vec{a}}{\left|\vec{a}\right|}$ and is denoted by $\hat{a}$, i.e. $\vec{a} = \left|\vec{a}\right|\hat{a}$.

2. Two or more vectors are said to be **equal** if they have the same magnitude and same direction.

3.
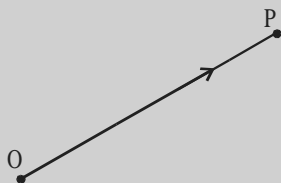
If initial point is chosen as origin O and P be any other point, then the **position** vector (p.v.) of P is $\overrightarrow{OP}$.

4. A vector having direction opposite to that of $\vec{a}$ but having the same magnitude is called the **negative** vector and is denoted by $-\vec{a}$.

5. A vector having its head and tail at the same point is called a **null** or a **zero** vector denoted by $\vec{0}$. e.g. $\overrightarrow{AA}$. A null vector is of zero magnitude and have any arbitrary direction.

6. **Vector addition :** If **a, b** are two vectors, then there exists three points A, B, C in the space such that $\mathbf{a} = \overrightarrow{AB}$ $\mathbf{b} = \overrightarrow{BC}$. Define $\mathbf{a} + \mathbf{b} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

7. **Scalar multiplication :** Let m be a scalar and **a** be a vector. Then the product $m\mathbf{a}$ is defined as follows.

  i) If $m > 0$, then $m\,\mathbf{a}$ is a vector of length $m\,|\mathbf{a}|$ and having direction same as to that of **a**.

  ii) If $m < 0$, then $m\,\mathbf{a}$ is a vector of length $(-m)\,|\mathbf{a}|$ and having direction opposite to that of **a**

  iii) If $m = 0$, then $m\,\mathbf{a} = 0$

8. Two nonzero vectors **a**, **b** are like vectors $\Leftrightarrow (\mathbf{a}, \mathbf{b}) = 0$, where (a, b) is the angle between vector a and vector b.

9. Two nonzero vectors **a**, **b** are unlike vectors $\Leftrightarrow (\mathbf{a}, \mathbf{b}) = 180°$

10. Two nonzero vectors **a**, **b** are parallel $\Leftrightarrow (\mathbf{a}, \mathbf{b}) = 0$ or $(\mathbf{a}, \mathbf{b}) = 180°$

11. If $\mathbf{r}_1 = x_1\,\mathbf{a} + y_1\,\mathbf{b} + z_1\,\mathbf{c}$, $r_2 = x_2\,\mathbf{a} + y_2\,\mathbf{b} + z_2\,\mathbf{c}$ then

$$\mathbf{r}_1 + \mathbf{r}_2 = (x_1 + x_2)\mathbf{a} + (y_1 + y_2)\mathbf{b} + (z_1 + z_2)\mathbf{c}$$

12. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$

13. If the position vectors of the points, P, Q are $x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k}$ and $x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k}$ then $|\overrightarrow{PQ}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

14. The vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $b = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $c = c_1 i + c_2\mathbf{j} + c_3\mathbf{k}$ are linearly dependent iff $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

15. The vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $b = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $c = c_1 i + c_2\mathbf{j} + c_3\mathbf{k}$ are

coplanar iff $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

16. Let $\mathbf{a}$, $\mathbf{b}$ be the position vectors of the points A, B respectively. The position vector of the point P which divides $\overline{AB}$ in the ratio $m : n$ is

$$\frac{m\mathbf{b} + n\mathbf{a}}{m + n}$$

17. Let $\mathbf{a}$, $\mathbf{b}$ be the position vectors of the points A, B respectively. The position vector of the point P which divides $\overline{AB}$ externally in the ratio

$m : n$ is $\dfrac{m\mathbf{b} - n\mathbf{a}}{m - n}$

18. **Dot product :** Let $\mathbf{a}$, $\mathbf{b}$ be two vectors. The *dot product* or *direct product* or *inner product* or *scalar product* of $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \cdot \mathbf{b}$ and is defined as follows

1) If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$

2) If $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\, |\mathbf{b}|\, \cos(\mathbf{a}, \mathbf{b})$

19. If $\mathbf{a}, \mathbf{b}$ are two vectors then $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

20. If a is a vector then $a \cdot a = |\mathbf{a}|^2$.

21. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three vectors, then
i) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
ii) $(\mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a}$.

22. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

23. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

24. If a, b are two nonzero vectors then $\cos(a, b) = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

25. **Cross product :** Let **a, b** be two vectors. The *cross product* or *vector product* or *skew product* of the vectors **a, b** is denoted by **a × b** and is defined as follows

   1) If **a = 0** or **b = 0** or a, b are parallel, then **a × b** = 0

   2) If **a ≠ 0, b ≠ 0, a, b** are not parallel, then **a × b** = $|\mathbf{a}||\mathbf{b}|$ sin **(a, b) n**, where **n** is the unit vector perpendicular to the planed generated by **a, b** so that **a, b, n** form a right handed system.

26. If **a, b** are two vectors then **a × b = – b × a.**

27. If **a, b, c** are three vectors, then
   i) **a × (b + c) = a × b + a × c**
   ii) **(b + c) × a = b × a + c × a**

28. i) **i × i = j × j = k × k = 0**
   ii) **i × j = k = – j × i, j × k = i = – k × j; k × i = j = – i × k.**

29. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

30. If a, b, c are the position vectors of the vertices of a triangle, then the vector area of the triangle = $\dfrac{1}{2}$ **(a × b + b × c + c × a)**.

31. The length of the projection of **b** on a vector perpendicular to **a** in the plane generated by **a, b** is $\dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$

32. If **a, b, c** are three vectors, then **(a × b) . c, a . (b × c)** are called **scalar triple** products of **a, b, c**.

33. If **a, b, c,** are vectors, then **(a × b) . c = (b × c) . a = (c × a) . b.**

34. If **a, b, c** are three vectors then
   $$\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} b & c & a \end{bmatrix} = \begin{bmatrix} c & a & b \end{bmatrix}$$
   $$= -\begin{bmatrix} b & a & c \end{bmatrix} = -\begin{bmatrix} c & b & a \end{bmatrix} = -\begin{bmatrix} a & c & b \end{bmatrix}$$

35. If **a, b** are two vectors then $\begin{bmatrix} a & a & b \end{bmatrix} = \begin{bmatrix} a & b & a \end{bmatrix} = \begin{bmatrix} b & a & a \end{bmatrix} = 0$.

36. If $\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n}$, $\mathbf{b} = b_1\mathbf{l} + b_2\mathbf{m} + b_3\mathbf{n}$, $\mathbf{c} = c_1\mathbf{l} + c_2\mathbf{m} + c_3\mathbf{n}$ where **l, m, n** form a right handed system of noncoplanar vectors, then

   $$\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} l & m & n \end{bmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

37. Three vectors **a, b, c** are coplanar iff $\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} = 0$.

38. If **a, b, c, d** are four vectors then $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a}.\mathbf{c} & \mathbf{a}.\mathbf{d} \\ \mathbf{b}.\mathbf{c} & \mathbf{b}.\mathbf{d} \end{vmatrix}$

39. If **a, b, c, d** are four vectors, then
$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\,\mathbf{b}\,\mathbf{d}]\,\mathbf{c} - [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\,\mathbf{d} = [\mathbf{a}\,\mathbf{c}\,\mathbf{d}]\,\mathbf{b} - [\mathbf{b}\,\mathbf{c}\,\mathbf{d}]\,\mathbf{a}.$

40. $\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix} = 1$

41. $\begin{bmatrix} \mathbf{a}+\mathbf{b} & \mathbf{b}+\mathbf{c} & \mathbf{c}+\mathbf{a} \end{bmatrix} = 2\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix}$

42. $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) + \mathbf{j} \times (\mathbf{k} \times \mathbf{i}) + \mathbf{k} \times (\mathbf{i} \times \mathbf{j}) = 0$

43. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$

44. $\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k}) = 2\mathbf{a}$

45. $\begin{bmatrix} \mathbf{a} \times \mathbf{b} & \mathbf{b} \times \mathbf{c} & \mathbf{c} \times \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix}^2$

Solved Example

1. Find the position vectors of the points which divide the join of the

points $2\vec{a} + 3\vec{b}$ and $\vec{a} - 2\vec{b}$ internally and externally in the ratio 3 : 2.

a. $\dfrac{7}{5}\vec{a}, -\vec{a} - 12\vec{b}$　　　　　　b. $-\dfrac{7}{5}\vec{a}, \vec{a} + 12\vec{b}$

c. $\dfrac{8}{5}\vec{a} + \vec{b}, -4\vec{a} - 13\vec{b}$　　　　d. None of these

Sol. Let P and Q be the points with position vectors

$2\vec{a} + 3\vec{b}$ and $\vec{a} - 2\vec{b}$ respectively. Let R and S be the points
dividing PQ in the ratio 3 : 2 internally and externally respectively.
Then
Position vector of R.

$$= \frac{3\left(\vec{a} - 2\vec{b} \right) + 2\left(2\vec{a} + 3\vec{b} \right)}{3+2} = \frac{7}{5}\vec{a}$$

Position vector of $S = \dfrac{3\left(\vec{a} - 2\vec{b}\right) - 2\left(2\vec{a} + 3\vec{a}\right)}{3 - 2} = -\vec{a} - 12\vec{b}$

2. If the position vectors of A, B, C and D are

$\hat{i} + \hat{j} + \hat{k}, \ 2\hat{i} + 5\hat{j}, \ 3\hat{i} + 2\hat{j} - 3\hat{k}$

and $\hat{i} - 6\hat{j} - \hat{k}$, then angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is

a. 0   b. $\dfrac{\pi}{4}$ c. $\dfrac{\pi}{2}$ d. p

Sol. $\overrightarrow{AB} = (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 4\hat{j} - \hat{k}$.

$= (\hat{i} - 6\hat{j} - \hat{k}) - (3\hat{i} + 2\hat{j} - 3\hat{k})$

$= -2\hat{i} - 8\hat{j} + 2\hat{k}$. Angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is given by

$\cos\theta = \dfrac{(\hat{i} + 4\hat{j} - \hat{k}).(-2\hat{i} - 8\hat{j} + 2\hat{k})}{\sqrt{1 + 16 + 1}\sqrt{4 + 64 + 4}} = -1 \Rightarrow \theta = \pi.$

3. Given vectors are $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}, \ \vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$. Find the value of $[\vec{a} \ \vec{b} \ \vec{c}] =$

a. –37   b. –19   c. –7   d. 49

Sol. $[\vec{a} \ \vec{b} \ \vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = -7$

4. If the vectors $\vec{a} = -2\hat{i} - 2\hat{j} + 4\hat{k} \ \ \vec{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}$ and

$\vec{c} = 4\hat{i} - \lambda\hat{j} - 2\hat{k}$ are coplanar, then the value of λ is

a. 2   b. –2   c. 4   d. –4

Sol. $\because$ Given vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

$\therefore [\vec{a} \ \vec{b} \ \vec{c}] = 0$

$\Rightarrow \begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -\lambda & -2 \end{vmatrix} = 0$

$\Rightarrow -2(-8 - 2\lambda) + 2(4 + 8) + 4(2\lambda - 16) = 0$

$\Rightarrow \lambda = 2$

5. If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$, then $\vec{a} + p\vec{c}$ is perpendicular to $\vec{b}$ if p =

    a. –5   b. 5   c. 8   d. 6

Sol: $\because \vec{a} + p\vec{c}$ is perpendicular to $\vec{b}$ $\therefore \left(\vec{a} + p\vec{c} \right).\vec{b} = 0$

$$\Rightarrow \vec{a}.\vec{b} + p\left(\vec{c}.\vec{b} \right) = 0$$

$$\Rightarrow p = -\frac{\vec{a}.\vec{b}}{\vec{c}.\vec{b}}$$

$$= -\frac{(-2+4+3)}{(-3+2)} = 5$$

6. For any three vectors $\vec{a}, \vec{b}, \vec{c} \cdot [\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]$ is equal to

    a. $2(\vec{a}+\vec{b}+\vec{c})$ b. $[\vec{a} \ \vec{b} \ \vec{c}]^2$

    c. $2[\vec{a} \ \vec{b} \ \vec{c}]$ d. None of these

Sol: $[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]$

$$= (\vec{a}+\vec{b}) \cdot [(\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})]$$

$$= (\vec{a}+\vec{b})[\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}]$$

$$= (\vec{a}+\vec{b}) \cdot [\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + 0 + \vec{c} \times \vec{a}]$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a})$$

$$+ \vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$= [\vec{a} \ \vec{b} \ \vec{c}] + 0 + 0 + 0 + 0 + [\vec{b} \ \vec{c} \ \vec{a}] = 2[\vec{a} \ \vec{b} \ \vec{c}]$$

7. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors, and $\vec{p}, \vec{q}$ and $\vec{r}$ be the vectors defined by the relations

$$\vec{p} = \frac{\vec{b} \times \vec{c}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}, \ \vec{q} = \frac{\vec{c} \times \vec{a}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} \text{ and } \vec{r} = \frac{\vec{a} \times \vec{b}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}, \text{ then}$$

$(\vec{a} + \vec{b}) \cdot \vec{p} + (\vec{b} + \vec{c}) \cdot \vec{q} + (\vec{c} + \vec{a}) \cdot \vec{r}$ is equal to

a. 0   b. 1   c. 2   d. 3

Sol. $\vec{a} \cdot \vec{p} = \vec{a} \cdot \dfrac{\left(\vec{b} \times \vec{c}\right)}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} = \dfrac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} \ \vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = 1$ and

$\vec{a} \cdot \vec{q} = \vec{a} \cdot \dfrac{\vec{c} \times \vec{a}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} = \dfrac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} \ \vec{a} \cdot \left(\vec{c} \times \vec{a}\right) = 0.$

Similarly, $\vec{b} \cdot \vec{q} = \vec{c} \cdot \vec{r} = 1$ and

$\vec{a} \cdot \vec{r} = \vec{b} \cdot \vec{p} = \vec{c} \cdot \vec{q} = \vec{c} \cdot \vec{p} = \vec{b} \cdot \vec{r} = 0.$

Therefore,

$\left(\vec{a} + \vec{b}\right) \cdot \vec{p} + \left(\vec{b} + c\right) \cdot \vec{q} + \left(\vec{c} + \vec{a}\right) \cdot \vec{r}$

$= \vec{a} \cdot \vec{p} + \vec{b} \cdot \vec{p} + \vec{b} \cdot \vec{q} + \vec{c} \cdot \vec{q} + \vec{c} \cdot \vec{r} + \vec{a} \cdot \vec{r} = 1 + 1 + 1 = 3.$

8 Complex Numbers

Definition

Any number of the form $a + ib$, where a and b are real numbers, is called complex number. Generally, complex numbers are denoted by z ($= a + ib$ say). Here a is called the real part of z, and b is the imaginary part of z. Notation for the same is Re $(z) = a$, Im $(z) = b$, where Re (z) stands for the real part of z and Im (z) stands for the imaginary part of z. Any complex number $a + ib$ can also be written as (a, b).

Properties

1. Two complex numbers z_1 and z_2 are **equal** if Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2). Therefore, $a + ib = c + id \Leftrightarrow a = c$ and $b = d$. Note that we cannot compare two complex numbers (except equality), i.e. we cannot say whether a complex number z_1 is more than or less an another complex number z_2.

2. Let $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2) \in C$. Then define the sum of z_1 and z_2 as $z_1 + z_2 = (a_1 + a_2, b_1 + b_2)$. Here $+$ is called **complex addition.**
For e.g. $(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$
Similarly, $(a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$

3. Let $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2) \in C$.
Then define the sum of z_1 and z_2 as $z_1 . z_2 = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1)$. Here '.' is called **complex multiplication.**
For e.g. $(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$

4. **Commutative law for addition:** If $z_1, z_2 \in C$, then $z_1 + z_2 = z_2 + z_1$

5. **Associative law for addition**: If $z_1, z_2, z_3 \in C$, then $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

6. **Distributive law for addition:** If $z_1, z_2, z_3 \in C$, then $z_1 + (z_2 + z_3) = z_1z_2 + z_1z_3$

Modulus and Argument

If P is a point in the complex plane corresponding to the complex number $z = a + ib$, then $a = r \cos\theta$, $b = r \sin\theta$, where $r = \sqrt{a^2 + b^2} = OP$ and $\theta = \tan^{-1}\dfrac{b}{a}$. r is called modulus of the complex number z and is written $|z|$ and q is called argument or amplitude of z, written as arg(z) or amp(z). Therefore, $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \arg(z) = \tan^{-1}\dfrac{b}{a}$, i.e. polar coordinates of z is (r, θ).

Euler's form

$z = a + ib = r(\cos\theta + i\sin\theta) = r.e^{i\theta}$, where $\cos\theta + i\sin\theta = e^{i\theta}$

7. $a - ib$ is called the **conjugate** of $z = a + ib$ and is denoted by $\bar{z}$.

Properties of Conjugate

(i) $z\bar{z} = |z|^2$

(ii) $\overline{z_1 \pm z_2} = \bar{z_1} \pm \bar{z_2}$

(iii) $\overline{z_1 z_2} = \bar{z_1}\,\bar{z_2}$

(iv) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z_1}}{\bar{z_2}}$

8.   If z_1 and z_2 be two complex numbers, then

(i) $|z_1 + z_2| \le |z_1| + |z_2|$

(ii) $|z_1 - z_2| \le |z_1| + |z_2|$

(iii) $|z_1 - z_2| \ge |z_1| - |z_2|$

(iv) $|z_1.z_2| \ge |z_1||z_2|$

(v) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ if $z_2 \ne 0$

(vi) $|kz_1| = k|z_1|$ if k is any positive real number.

Cube Roots of Unity

If z is a cube root of unity, then

$z = 1^{\frac{1}{3}} = 1(\cos 0 + i\sin 0)^{\frac{1}{3}} = (\cos 2r\pi + i\sin 2r\pi)^{\frac{1}{3}}$

$= \cos\dfrac{2r\pi}{3} + i\,\sin\dfrac{2r\pi}{3}$, where r = 0, 1, 2

$\therefore z = 1,\ \cos\dfrac{2\pi}{3} + i\,\sin\dfrac{2\pi}{3}, \cos\dfrac{4\pi}{3} + i\,\sin\dfrac{4\pi}{3}$ or $z = 1, \dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$,

which are the cube roots of unity. They are also denoted by 1, ω, ω^2,

where $\omega = \dfrac{-1+i\sqrt{3}}{2}$.

Properties of Cube-Roots of Unity

9. $1 + \omega + \omega^2 = 0$

10. $\omega^3 = 1$

11.   The product of the cube roots of unity is one.

12.   1, ω, ω^2 are in Geometric Progression.

Solved Examples

1. $\left(\dfrac{i+\sqrt{3}}{-i+\sqrt{3}}\right)^{200}+\left(\dfrac{i-\sqrt{3}}{i+\sqrt{3}}\right)^{200}=$

    a. 1   b. –1   c. 0   d. None of these

Sol. We have $\left(\dfrac{i+\sqrt{3}}{-i+\sqrt{3}}\right)^{200}+\left(\dfrac{i-\sqrt{3}}{i+\sqrt{3}}\right)^{200}$

$$=\left\{\frac{i(i+\sqrt{3})}{i(-i+\sqrt{3})}\right\}^{200}+\left\{\frac{i(i-\sqrt{3})}{i(i+\sqrt{3})}\right\}^{200}$$

$$=\left\{\frac{\left(\dfrac{-1+i\sqrt{3}}{2}\right)}{-\left(\dfrac{-1-i\sqrt{3}}{2}\right)}\right\}^{200}+\left\{\frac{\left(\dfrac{-1-i\sqrt{3}}{2}\right)}{\left(\dfrac{-1+i\sqrt{3}}{2}\right)}\right\}^{200}$$

$$=\left(\frac{\omega}{-\omega^2}\right)^{200}+\left(\frac{\omega^2}{\omega}\right)^{200}$$

$$\left[\because \omega=\frac{-1+i\sqrt{3}}{2}\ \text{and}\ \omega^2=\frac{-1-i\sqrt{3}}{2}\right]$$

$$=\left(-\frac{1}{\omega}\right)^{200}+\omega^{200}=(-\omega^2)^{200}+\omega^{200}\quad\left[\because \frac{1}{\omega}=\omega^2\right]$$

$$=\omega^{400}+\omega^{200}=(\omega^3)^{133}\,\omega+(\omega^3)^{66}\,\omega^2=\omega+\omega^2=-1$$

2. $\left(\dfrac{a+b\omega+c\omega^2+d\omega^3+e\omega^4}{a\omega+b\omega^2+c\omega^3+d\omega^4+e}\right)^5=$ (Given $\omega^5=1$)

    a. ω b. ω^3 c. ω^5 d. None of these

Sol. Let $Z=\left(\dfrac{a+b\omega+c\omega^2+d\omega^3+e\omega^4}{a\omega+b\omega^2+c\omega^3+d\omega^4+e}\right)$

$$=\left(\frac{a+b\omega+c\omega^2+d\omega^3+e\omega^4}{a\omega+b\omega^2+c\omega^3+d\omega^4+e\omega^5}\right)$$

Since $\omega^5=1$,

$$Z=\frac{1}{\omega}\Rightarrow Z^5=\frac{1}{\omega^5}=1$$

9 Coordinate Geometry

Cartesian Coordinates
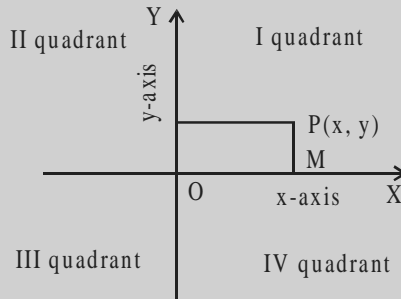
Let OX and OY be two mutually perpendicular straight lines in the plane of paper. The line OX is called x-axis, the line OY is y-axis, whereas the two together are called the **coordinate axes**. The plane is called the XY-plane or coordinate plane. The point 'O' is known as **origin**. From any point 'P' in the plane, draw a straight line parallel to OY to meet OX in M.

The distance OM (= x) is called the **abscissa**, and the distance MP (= y) is called the **ordinate** of the point P, whereas the abscissa and the ordinate together are its coordinates, denoted simply as P(x, y). For an arbitrary pair of real numbers x and y, there exists a unique point A in the XY-plane for which x will be its abscissa and y its ordinate. Note that (x, y) is an ordered pair, which means that point (x, y) is different from (y, x).

Some Standard Properties and Results

1. The distance between the points A (x_1, y_1) and B (x_2, y_2) is

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

2. Three or more points are said to be **collinear** if they lie on a line.

3. If A, B and C are collinear, then AB + BC = AC or AC + CB = AB or BA + AC = BC.

4. The point which divides the line segment joining the points A (x_1, y_1), B (x_2, y_2) in the ratio $l : m$

(i) internally is $\left(\dfrac{lx_2 + mx_1}{l+m}, \dfrac{ly_2 + my_1}{l+m} \right)(l+m \neq 0)$

(ii) externally is $\left(\dfrac{lx_2 - mx_1}{l-m}, \dfrac{ly_2 - my_1}{l-m} \right)(l \neq m)$

5. The midpoint of the line segment joining A (x_1, y_1), B (x_2, y_2) is

$\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right)$

6. If P (x, y) lies in the line joining A (x_1, y_1), B (x_2, y_2) then

$\dfrac{x_1 - x}{x - x_2} = \dfrac{y_1 - y}{y - y_2}$ and P divides $\overline{AB}$ in the ratio $x_1 - x : x - x_2$ that is

also equals to

$y_1 - y : y - y_2$.

7. If (x_1, y_1), (x_2, y_2), (x_3, y_3) are three consecutive vertices of a parallelogram, then the fourth vertex is $(x_1 - x_2 + x_3, y_1 - y_2 + y_3)$.

8. The centroid of the triangle formed by the points A (x_1, y_1), B (x_2, y_2),

C (x_3, y_3) is. $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3} \right)$

9. The area of the triangle formed by the points A (x_1, y_1) and B (x_2, y_2)

and C (x_3, y_3) is $= \dfrac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \dfrac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$

$= \dfrac{1}{2} \left[(x_1 y_2 - x_2 y_1 + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) \right]$

10. The area of the triangle formed by the points O $(0,0)$, A $(x1, y1)$, B $(x2, y2)$ is $\dfrac{1}{2} |x_1 y_2 - x_2 y_1|$

11. The area of the triangle formed by the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_1 - x_3 \\ y_1 - y_2 & y_1 - y_3 \end{vmatrix}$$

12. Three points A, B, C are collinear if area of Δ ABC is zero.

13. The incenre of ΔABC with sides a, b and c and vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right).$$

14. Circumcentre of triangle $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by

$$O\left[\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right]$$

15. Orthocenter of triangle $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by

$$H\left[\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 + y_3 \tan C}{\tan A + \tan B + \tan C} \right]$$

Solved Examples

1. The points of trisection of the line joining the points (0, 3) and (6, –3) are

a. (2, 0); (4, –1)   b. (2, –1); (4, 1)

c. (3, 1) ;(4,–1)   d. (2, 1); (4, –1)

Sol: Let P, Q be the points of trisection

$$\therefore \quad P \equiv \left(\frac{0+6}{3}, \frac{6-3}{3} \right) \text{or} (2,\ 1)\ ;$$

$$\therefore \quad Q \equiv \left(\frac{0+12}{3}, \frac{3\text{-}6}{3} \right) \text{or} (4, -1) \qquad .$$

2. A line is of length 10 units and one end is at the point (2, –3); if the abscissa of the other end be 10, then ordinate must be

a. 3 or –9   b. 9 or –3

c. 4   d. 6

Sol: Let the ordinate be y. Then, $\sqrt{(10-2)^2 + (y+3)^2} = 10$.

Therefore, y must be 3 or –9.

3. The length of the side of an equilateral triangle with vertex (1, 1) and

circumcentre $\left(\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}} \right)$ is

a. $\sqrt{18}$ b. $\sqrt{8}$ c. $\sqrt{28}$ d. None of these

Sol: Length of side of an equilateral triangle is equal to $\sqrt{3}$ R where R is the circumradius.

$$R = \sqrt{\left(1 - \dfrac{1}{\sqrt{3}}\right)^2 + \left(1 + \dfrac{1}{\sqrt{3}}\right)^2} = \sqrt{\dfrac{8}{3}}.$$

Therefore, side $= \sqrt{3} \times \sqrt{\dfrac{8}{3}} = \sqrt{8}$

4. The orthocentre of the triangle whose vertices are (0,0), (3, $\sqrt{3}$), (0, 2$\sqrt{3}$) is

a. (0, 0)   b. ($\sqrt{3}$, 0)   c. (1, $\sqrt{3}$)   d. None of these

Sol: Since the triangle is equilateral, orthocentre is same as centroid G =

$$\left(\dfrac{0 + 3 + 0}{3}, \dfrac{0 + \sqrt{3} + 2\sqrt{3}}{3} \right) = \left(1, \sqrt{3}\right).$$

5. If the line joining (c + 1, 2c) and (4c, –3c) is perpendicular to the line joining (3c, 1 – c) and (–c, –4c) then c is equal to

a. 3   b. –3   c. $\dfrac{1}{3}$ d. $-\dfrac{1}{3}$

Sol: Product of slopes $= -1$ i.e. $\dfrac{-5c}{3c - 1} \times \dfrac{-3c - 1}{-4c} = -1$ or $-15c - 5 = -12c + 4$

or c = –3.

6. If the mid-points of sides of a triangle are (2, 1), (0, –3) and (4, 5), then the coordinates of the centroid of the triangle are

a. (2, 1)   b. (0, 2)

c. (1, 3)   d. Cannot be determined

Sol: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right) \equiv (2,1) \Rightarrow x_1 + x_2 = 4, \ y_1 + y_2 = 2$

Similarly, $x_2 + x_3 = 0$, $y_2 + y_3 = -6$, and $x_3 + x_1 = 8$, $y_3 + y_1 = 10$

Hence, $\dfrac{x_1 + x_2 + x_3}{3} = \dfrac{4 + 0 + 8}{2.3} = 2$ and $\dfrac{y_1 + y_2 + y_3}{3} = \dfrac{2 - 6 + 10}{2.3} = 1$

i.e. centroid is (2,1).

Short cut: Centroid of the triangle formed by the mid-points of sides will be same as that formed by the vertices.

10 Trigonometry

Some Standard Formulae and Results

1. $\sin^2\theta + \cos^2\theta = 1$, $\sin^2\theta = 1 - \cos^2\theta$, $\cos^2\theta = 1 - \sin^2\theta$

2. $1 + \tan^2\theta = \sec^2\theta$, $\tan^2\theta = \sec^2\theta - 1$, $\sec^2\theta - \tan^2\theta = 1$

3. $1 + \cot^2\theta = \csc^2\theta$, $\cot^2\theta = \csc^2\theta - 1$, $\csc^2\theta - \cot^2\theta = 1$

4. $\sec\theta + \tan\theta = \dfrac{1}{\sec\theta - \tan\theta}$

5. $\csc\theta + \cot\theta = \dfrac{1}{\csc\theta - \cot\theta}$

6. (i) $\cos(A + B) = \cos A \cos B - \sin A \sin B$
(ii) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

7. (i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$
(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

8. (i) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(ii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

9. $\sin(A + B) \cdot \sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$.

10. $\cos(A + B) \cdot \cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

11. (i) $\tan\left(\dfrac{\pi}{4} + A\right) = \dfrac{1 + \tan A}{1 - \tan A} = \dfrac{\cos A + \sin A}{\cos A - \sin A}$

(ii) $\tan\left(\dfrac{\pi}{4} - A\right) = \dfrac{1 - \tan A}{1 + \tan A} = \dfrac{\cos A - \sin A}{\cos A + \sin A}$

12. (i) $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$
(ii) $\sin(A + B) - \sin(A - B) = 2\cos A \sin B$
(iii) $\cos(A + B) + \cos(A - B) = 2\cos A \cos B$
(iv) $\cos(A - B) - \cos(A + B) = 2\sin A \sin B$

13. (i) $\sin C + \sin D = 2 \sin \left(\dfrac{C+D}{2} \right) \cos \left(\dfrac{C-D}{2} \right)$

(ii) $\sin C - \sin D = 2 \cos \left(\dfrac{C+D}{2} \right) \sin \left(\dfrac{C-D}{2} \right)$

(iii) $\cos C + \cos D = 2 \cos \left(\dfrac{C+D}{2} \right) \cdot \cos \left(\dfrac{C-D}{2} \right)$

(iv) $\cos C - \cos D = -2 \sin \left(\dfrac{C+D}{2} \right) \cdot \sin \left(\dfrac{C-D}{2} \right)$

$$= 2 \sin \left(\dfrac{C+D}{2} \right) \cdot \sin \left(\dfrac{D-C}{2} \right)$$

Notations Used: We shall denote the lengths of sides BC, CA and AB of a ΔABC by a, b and c. Semi-perimeter of the triangle will be denoted by

$s = \left(\dfrac{a+b+c}{2} \right)$ and its area by S or Δ.

Circumradius, inradius and 3 exradii will be denoted by R, r and (r_1, r_2, r_3) respectively.

14. **Law of Sines:** $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{1}{2R}$

15. **Law of Cosines:**

$$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \dfrac{c^2 + a^2 - b^2}{2ca}, \quad \cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$$

16. $\sin \dfrac{A}{2} = \sqrt{\left[\dfrac{(s-b)(s-c)}{bc} \right]}, \quad \sin \dfrac{B}{2} = \sqrt{\left[\dfrac{(s-c)(s-a)}{ca} \right]},$

$\sin \dfrac{C}{2} = \sqrt{\left[\dfrac{(s-a)(s-b)}{ab} \right]}$

17. $\cos \dfrac{A}{2} = \sqrt{\left[\dfrac{s(s-a)}{bc} \right]}, \quad \cos \dfrac{B}{2} = \sqrt{\left[\dfrac{s(s-b)}{ca} \right]}, \quad \cos \dfrac{C}{2} = \sqrt{\left[\dfrac{s(s-c)}{ab} \right]}$

18. $\tan \dfrac{A}{2} = \sqrt{\left[\dfrac{(s-b)(s-c)}{s(s-a)} \right]}, \quad \tan \dfrac{B}{2} = \sqrt{\left[\dfrac{(s-c)(s-a)}{s(s-b)} \right]},$

$\tan \dfrac{C}{2} = \sqrt{\left[\dfrac{(s-a)(s-b)}{s(s-c)} \right]}.$

19. $\cot \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}}$, $\cot \dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{(s-a)(s-c)}}$, $\cot \dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}}$

20. **Area of a triangle:** (i) $\Delta = \dfrac{1}{2} ab \sin C = \dfrac{1}{2} bc \sin A = \dfrac{1}{2} ca \sin B$

(ii) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

(iii) $\Delta = \dfrac{abc}{4R} = rs$.

21. **Projection formula:** $a = b \cos C + c \cos B$. Likewise for b and c.

22. **Napier's analogy:** (i) $\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}$

(ii) $\tan \left(\dfrac{C-A}{2} \right) = \dfrac{c-a}{c+a} \cot \dfrac{B}{2}$

(iii) $\tan \left(\dfrac{A-B}{2} \right) = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$

23. **Circumradius**

(i) $R = \dfrac{a}{2 \sin A} = \dfrac{b}{2 \sin B} = \dfrac{c}{2 \sin C}$

(ii) $R = \dfrac{abc}{4\Delta}$

24. **Inradius**

(i) $r = \dfrac{\Delta}{s}$

(ii) $r = (s-a) \tan \dfrac{A}{2}, r = (s-b) \tan \dfrac{B}{2}, r = (s-c) \tan \dfrac{C}{2}$

(iii) $r = 4R \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$

25. **Radii of escribed circles**

(i) $r_1 = 4R \sin \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$, $r_2 = 4R \cos \dfrac{A}{2} \sin \dfrac{B}{2} \cos \dfrac{C}{2}$,

$r_3 = 4R \cos \dfrac{A}{2} \cos \dfrac{B}{2} \sin \dfrac{C}{2}$

(ii) $r_1 = \dfrac{\Delta}{s-a} = s \tan \dfrac{A}{2}, r_2 = \dfrac{\Delta}{s-b} = s \tan \dfrac{B}{2}, r_3 = \dfrac{\Delta}{s-c} = s \tan \dfrac{C}{2}$

Trigonometric Equations

Some Standard Results

1. Solution set of $\sin x = 0$ is $\{n\pi : n \in Z\}$.

2. Solution set of $\cos x = 0$ is $\left\{(2n+1)\dfrac{\pi}{2} : n \in Z\right\}$

3. Solution set of $\tan x = 0$ is $\{n\pi : n \in Z\}$.

4. Solution set of $\sin x = k$, $k \in R$, $|k| \le 1$ is $\{n\pi + (-1)^n \alpha : n \in Z\}$, where α is the principal solution.

5. Solution set of $\cos x = k$, $k \in R$, $|k| \le 1$ is $\{(2n\pi \pm \alpha) : n \in Z\}$, where α is the principal solution.

6. Solution set of $\tan x = k$, $k \in R$ is $\{n\pi + \alpha : n \in Z\}$, where α is the principal solution.

7. Solution set of $\sin^2 x = \sin^2 \alpha$ is $\{n\pi \pm \alpha : n \in Z\}$, where α is the principal solution of $\sin x = \sin \alpha$.

8. Solution set of $\cos^2 x = \cos^2 \alpha$ is $\{n\pi \pm \alpha : n \in Z\}$, where α is the principal solution of $\cos x = \cos \alpha$.

9. Solution set of $\tan^2 x = \tan^2 \alpha$ is $\{n\pi \pm \alpha : n \in Z\}$, where α is the principal solution of $\tan x = \tan \alpha$.

Inverse Circular Functions

Some Standard Results

1. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$, for $x > 0$, $y > 0$, $xy < 1$

$= \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$, for $x > 0$, $y > 0$, $xy > 1$.

2. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$ for $x < 0$, $y < 0$, $xy < 1$

$= -\pi + \tan^{-1}\dfrac{x+y}{1-xy}$ for $x < 0$, $y < 0$, $xy > 1$

3. If $x > 0$, $y > 0$ then $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \dfrac{x-y}{1+xy}$

4. If $x < 0$, $y < 0$ then $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\dfrac{x-y}{1+xy} \right)$.

5. (i) $2 \sin^{-1} x = \sin^{-1} \left(2x\sqrt{1-x^2} \right)$, for $x \leq \dfrac{1}{\sqrt{2}}$

(ii) $2 \sin^{-1} x = \pi - \sin^{-1} \left(2x\sqrt{1-x^2} \right)$, for $x > \dfrac{1}{\sqrt{2}}$

6. (i) $2 \cos^{-1} x = \cos^{-1} (2x^2 - 1)$, for $x \leq \dfrac{1}{\sqrt{2}}$

(ii) $2 \cos^{-1} x = \pi - \cos^{-1} (1 - 2x^2)$, for $x > \dfrac{1}{\sqrt{2}}$

7. $2 \tan^{-1} x = \tan^{-1} \left(\dfrac{2x}{1-x^2} \right)$, if $x^2 < 1$

$= \pi + \tan^{-1} \left(\dfrac{2x}{1-x^2} \right)$, if $x^2 > 1$

8. $2 \tan^{-1} x = \sin^{-1} \dfrac{2x}{1+x^2}$, $\forall \; x \in R$

$= \cos^{-1} \dfrac{1-x^2}{1+x^2}$, if $x > 0$

$= -\cos^{-1} \dfrac{1-x^2}{1+x^2}$, if $x < 0$

9. (i) $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$, for $0 \leq x \leq 1$

(ii) $\sin^{-1} x = - \cos^{-1} \sqrt{1-x^2}$, for $-1 \leq x < 0$

10. (i) $\cos^{-1} x = \sin^{-1} \sqrt{1-x^2}$, for $0 \leq x \leq 0$

(ii) $\cos^{-1} x = \pi - \sin^{-1} \sqrt{1-x^2}$, for $-1 \leq x < 0$

Angle of Elevation and Depression of a Point
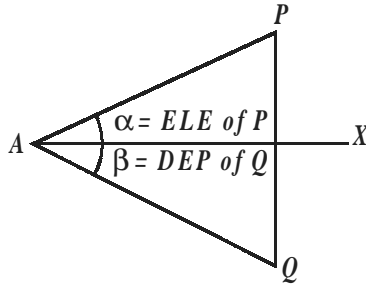
Suppose a straight line AX is drawn in the horizontal direction. Then the angle XAP where P is a point above AX is called **angle of elevation** of P as seen from A. Similarly, the angle XAQ, where Q is below AX, is called **angle of depression** of some point Q.

Note that both the angles — angle of elevation and angle of depression — are measured with the horizontal line. In the figure shown, the angle of elevation of top of the pole AC standing on inclined plane PQ is ∠ABM (not ∠ABC).

Solved Examples

1. If $\sec 4\theta - \sec 2\theta = 2,$ then the general value of θ is

a. $(2n+1)\dfrac{\pi}{4}$

b. $(2n+1)\dfrac{\pi}{10}$

c. $n\pi + \dfrac{\pi}{2}$ or $\dfrac{n\pi}{5} + \dfrac{\pi}{10}$

d. None of these

Sol. $\sec 4\theta - \sec 2\theta = 2 \Rightarrow \cos 2\theta - \cos 4\theta = 2\cos 4\theta \ \cos 2\theta$

$\Rightarrow -\cos 4\theta = \cos 6\theta$

$\Rightarrow 2\cos 5\theta \ \cos\theta = 0$

$\Rightarrow \theta = n\pi + \dfrac{\pi}{2}$ or $\dfrac{n\pi}{5} + \dfrac{\pi}{10}.$

2. $\left(1 + \cos\dfrac{\pi}{8}\right)\left(1 + \cos\dfrac{3\pi}{8}\right)\left(1 + \cos\dfrac{5\pi}{8}\right)\left(1 + \cos\dfrac{7\pi}{8}\right)$ is equal to

a. $\dfrac{1}{2}$

b. $\dfrac{1}{8}$

c. $\cos\dfrac{\pi}{8}$

d. $\dfrac{1+\sqrt{2}}{2\sqrt{2}}$

Sol. $\left(1 + \cos\dfrac{\pi}{8}\right)\left(1 + \cos\dfrac{3\pi}{8}\right)\left(1 - \cos\dfrac{3\pi}{8}\right)\left(1 - \cos\dfrac{\pi}{8}\right)$

$$= \left(1 - \cos^2 \frac{\pi}{8}\right)\left(1 - \cos^2 \frac{3\pi}{8}\right)$$

$$= \sin^2 \frac{\pi}{8} \sin^2 \frac{3\pi}{8} = \frac{1}{4}\left(2 \sin \frac{\pi}{8} \sin \frac{3\pi}{8}\right)^2$$

$$= \frac{1}{4}\left(\cos \frac{\pi}{4} - \cos \frac{\pi}{2}\right)^2 = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \theta\right)^2 = \frac{1}{8}$$

3. In $\triangle ABC$, $a = \sqrt{3} + 1$, $B = 30°$, $C = 45°$, then $C =$

   a. 1   b. 2   c. 3   d. 4

Sol. $B = 30°$, $C = 45° \Rightarrow C = 105°$

$$\because \quad c = \frac{a \sin C}{\sin A} \quad \Rightarrow \quad \frac{\left(\sqrt{3} + 1\right) \sin 45°}{\sin 75°}$$

$$\Rightarrow \quad \frac{\left(\sqrt{3} + 1\right) . \dfrac{1}{\sqrt{2}}}{\dfrac{\left(\sqrt{3} + 1\right)}{2\sqrt{2}}} = 2$$

4. In a $\triangle ABC$, if $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$ and the side $a = 2$, then area of the triangle is

   a. 1   b. 2   c. $\dfrac{\sqrt{3}}{2}$ d. $\sqrt{3}$

Sol. $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$

$$\Rightarrow \quad \frac{\cos A}{k \sin A} = \frac{\cos B}{k \sin B} = \frac{\cos C}{k \sin C}$$

$$\Rightarrow \quad \cot A = \cot B = \cot C$$

$$\Rightarrow \quad A = B = C = 60°$$

$\therefore \quad \triangle ABC$ is equilateral

$$\therefore \text{Area} = \frac{\sqrt{3}}{4} \times (2)^2 = \sqrt{3} \text{ sq. units}$$

5.   Find the maximum and minimum values of 6 sinx cos x + 4xcos2x.

Sol.   We have 6 sinx cosx + 4 cos2x = 3 sin2x + 4 cos2x. Therefore, the maximum and minimum values of 3 sin2x + 4 cos2x are

$$\sqrt{\left(3^2 + 4^2\right)} \text{ and } -\sqrt{\left(3^2 + 4^2\right)}, \text{ i.e. 5 and } -5 \text{ respectively.}$$

6.   If $\sin\theta + \text{cosec}\theta = 2$, then $\sin^2\theta + \text{cosec}^2\theta =$ is equal to
a. 1   b 4   c. 2   d. None of these

Sol. $\sin\theta + \cos ec\,\theta = 2 \Rightarrow \left(\sin\theta + \cos ec\,\theta\right)^2 = 4$

$\Rightarrow \sin^2\theta + \cos ec^2\theta + 2 = 4$

$\Rightarrow \sin^2\theta + \cos ec^2\theta = 2$

7.   If $3 \tan(\theta - 15°) = \tan(\theta + 15°)$, $0 < \theta < \pi$, then find the value of θ

Sol. $3\tan\left(\theta - 15^0\right) = \tan\left(\theta + 15^0\right)$

$\Rightarrow \dfrac{\tan\left(\theta + 15^0\right)}{\tan\left(\theta - 15^0\right)} = 3 \Rightarrow \dfrac{\tan(\theta + 15°) + \tan(\theta - 15°)}{\tan(\theta + 15°) - \tan(\theta - 15°)}$

$= \dfrac{3 + 1}{3 - 1} \Rightarrow \dfrac{\sin(\theta + 15° + \theta - 15°)}{\sin(\theta + 15° - \theta + 15°)} = 2$

$\Rightarrow \dfrac{\sin 2\theta}{\sin 30°} = 2 \Rightarrow \sin 2\theta = 1 \quad \Rightarrow 2\theta = \dfrac{\pi}{2} \ (\because 0 < \theta < \pi) \Rightarrow \theta = \dfrac{\pi}{4}.$

8.   In a Δ ABC, $\dfrac{a\cos A + b\cos B + c\cos C}{a + b + c}$ is equal to

a. $\dfrac{r}{R}$ b. $\dfrac{R}{r}$ c. $\dfrac{2r}{R}$ d. $\dfrac{R}{2r}$

Sol. $\dfrac{2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C}{2R\sin A + 2R\sin B + 2R\sin C} = \dfrac{\sin 2A + \sin 2B + \sin 2C}{2(\sin A + \sin B + \sin C)}$

$= \dfrac{4\sin A\sin B\sin C}{2\left(4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}\right)} = 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{r}{R}$

9.   If r, r_1, r_2 and r_3 have their usual meanings, the value of $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$ is

a. 0   b. 1   c. $\dfrac{1}{r}$ d. None of these

Sol. $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{s-a}{\Delta} + \dfrac{s-b}{\Delta} + \dfrac{s-c}{\Delta} = \dfrac{3s-2s}{\Delta} = \dfrac{s}{\Delta} = \dfrac{1}{r}$

10. If two angles of a triangle are $\cot^{-1}2$ and $\cot^{-1}3$, then the third angle is   equal to

     a. $\dfrac{\pi}{6}$ b. $\dfrac{\pi}{4}$ c. $\dfrac{\pi}{3}$ d. $\dfrac{3\pi}{4}$

Sol. (d) Sum of two angles $= \cot^{-1}2 + \cot^{-1}3$

$= \tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{3}\right)$

$= \tan^{-1}\left(\dfrac{\dfrac{1}{2}+\dfrac{1}{3}}{1-\dfrac{1}{2}\times\dfrac{1}{3}}\right)$

$= \tan^{-1}1 = \dfrac{\pi}{4}$

Third angle $= \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

11. A man at top of the $30\sqrt{3}$ m high tower sees a car moving towards the tower at angle of depression of 30°. After some time the angle of depression becomes 60°. Find the distance travelled by car in that duration.

     a. 120   b. 90   c. 60   d. 30

Sol.
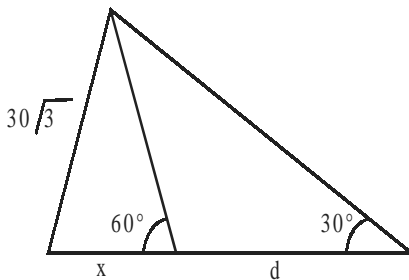
$\dfrac{30\sqrt{3}}{x} = \tan 60°$ and $\dfrac{30\sqrt{3}}{x+d} = \tan 30°$

$\Rightarrow \mathsf{x = 30}$ and $\dfrac{30\sqrt{3}}{30+d} = \dfrac{1}{\sqrt{3}}$, i.e.

$90 = 30 + d$, i.e. $d = 60$m

11 Permutation, Combination & Probability

Definition

Each of the different orders of arrangements, obtained by taking some, or all, of a number of things, is called a **Permutation**. Each of the different groups, or collections, that can be formed by taking some, or all, of a number of things, irrespective of the order in which the things appear in the group, is called a **Combination**.

Some Standard Properties and Results

1. The number of (linear) permutations that can be formed by taking r things at a time from a set of n dissimilar things ($r \leq n$) is denoted by ^nP_r or $P(n, r)$ or $P\begin{pmatrix} n \\ r \end{pmatrix}$.

2. The number of permutations of n dissimilar things taken r at a time is equal to the number of ways of filling of r blank places arranged in a row by n dissimilar things.

3. $^nP_r = n\,(n-1)\,(n-2)\,...\,(n-r+1)$.

4. If n is a nonnegative integer, then factorial n is denoted by n! or n! and defined as follows.
   (i) $0! = 1$
   (ii) If $n > 0$ then $n! = n \times (n-1)!$

5. If n is positive integer, then n! is the product of first n positive integers. i.e., $n! = 1 \cdot 2 \cdot 3 \ldots n$

6. $^nP_r = \dfrac{n!}{(n-r)!}$

7. The number of permutations of n dissimilar things taken all at a time $^nP_n = n!$.

8. $^nP_r = {}^{(n-1)}P_r + r\,{}^{(n-1)}P_{r-1}$

9. The number of permutations of n dissimilar things taken not more than r at a time, when each things may occur at any number of times is

$$\dfrac{n\left(n^r - 1\right)}{n-1}.$$

10. The number of permutations of n things taken all at a time when p of them are all alike and the rest all different is $\dfrac{n!}{p!}$.

11. The number of circular permutations of n different things taken r at a time is $\dfrac{^nP_r}{r}$.

12. The number of circular permutations of n different things taken all at a time is $(n-1)!$.

13. The number of circular permutations of n things taken r at a time in one direction is $\dfrac{^nP_r}{2r}$.

14. The number of circular permutations of n things taken all at a time in one direction is $\dfrac{1}{2}(n-1)!$.

15. A selection that can be formed by taking some or all of a finite set of things (or objects) is called a combination.

16. Formation of a combination by taking r elements from a finite set A means picking up an r element subset of A.

17. The number of combinations of n dissimilar things taken r at a time is equal to the number of r element subsets of a set containing n elements.

18. The number of combinations of n dissimilar things taken r at a time is denoted by ^nC_r or C(n, r) or $C\begin{pmatrix} n \\ r \end{pmatrix}$ or $\begin{pmatrix} n \\ r \end{pmatrix}$

19. $^nC_r = \dfrac{n!}{r!(n-r)!}$.

20. $^nC_r = \dfrac{^nP_r}{r!} = \dfrac{n(n-1)(n-2)....(n-r+1)}{1.2.3....r}$.

21. $^nC_r = {^nC_{n-r}}$

22. $^nC_r + {^nC_{r-1}} = {^{(n+1)}C_r}$

23. If $^nC_r = {^nC_s}$ then r = s or r + s = n.

24. The number of diagonals in a regular polygon of n sides is

$${}^nC_2 - n = \frac{n(n-3)}{2}.$$

25. The number of ways in which (m + n) things can be divided into two different groups of m and n things respectively is $\dfrac{(m+n)!}{m!\,n!}$.

26. The number of ways in which 2n things can be divided into two equal groups of n things each is $\dfrac{(2n)!}{2!(n!)^2}$.

27. The number of ways in which kn things can be divided into k equal groups of n things each is $\dfrac{(kn)!}{k!(n!)^k}$.

28. The total number of combinations of (p + q) things taken any number at a time when p things are alike of one kind and q things are alike of second kind is (p + 1) (q + 1).

29. The total number of combinations of p + q things taken any number at a time, includes the case in which nothing will be selected.

30. The total number of combinations of (p + q) things taken one or more at a time when p things are alike of one kind and q things are alike of second kind is (p + 1) (q + 1) – 1.

31. The total number of combinations of n different things taken any number at a time is 2^n.

32. The total number of combinations of n different things taken one or more at a time is $2^n - 1$.

33. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots {}^nC_n = 2^n$.

34. Any change in the given order of the things is called a **derangement.** In n things from an arrangement in a row, the number of ways in which they can be deranged so that none of them occupies its original place is

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \cdot \frac{1}{n!}\right).$$

Solved Example

1. Seven schools are participating in a Quiz competition. In how many ways can the first three prizes be won?

Sol. The total number of ways in which the first three prizes can be won is the number of arrangements of seven different things taken 3 at a time. So, the required number of ways

$$= {}^7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \times 6 \times 5 \times 4!}{4!} = 210$$

2. In how many ways 5 boys and 3 girls can be seated in a row so that no two girls are together?

Sol. The 5 boys can be selected in a row, i.e. ${}^5P_5 = 5!$ ways. In each of these arrangements 6 places are created shown by the marks as given below:
_B _ B _ B _B _ B _
Since no 2 girls have to sit together, so we may arrange 3 girls in 6 places. This can be done in ${}^6P_3$ ways, i.e. 3 girls can be seated in ways. Hence, the total number of sitting arrangements

$$= {}^5P_5 \times {}^6P_3 = 5! \times 6 \times 5 \times 4 = 14400$$

3. Let there be 10 boys and 6 girls in a class and we have to select a group of 6, if three particular boys never come and two particular girls are always present. Here, we have to select a group of 4 as 2 particular girls are already included in the group, from remaining 11 students (7 boys and 4 girls) which can be done in ${}^{11}C_4$ ways.

(a) The number of ways in which (m + n) things can be divided into two groups containing m and n things respectively $= {}^{m+n}C_n =$

$$\frac{(m+n)!}{m!\,n!} = {}^{m+n}C_m.$$

(b) If 2m things are to be divided into two groups, each containing m things, the number of ways $= \dfrac{(2m)!}{2(m!)^2}$

(c) The number of ways to divide n things into different groups, one containing p things, another q things and so on $= \dfrac{(p+q+r+....)!}{p!\,q!\,r!....}$
where $n = p + q + r + ...$

(d) If some or all of n things be taken at a time, then the number of combinations $= {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$.

(e) Total number of combinations of n things, taking some or all at a time, when p of them are alike of one kind, q of them are alike of another kind and so on is $[(p + 1)(q + 1)(r + 1)...] - 1$,
where $n = p + q + r + ...$

(f) Total number of ways to make a selection, by taking some or all of $p_1 + p_2 + ... + p_r$ things, where p1 are alike of one kind, p_2 are alike of second kind,..., pr are alike of rth kind is given by $(p_1 + 1) (p_2 + 1)...(p_r + 1) - 1$.

(g) If there are p_1 objects of one kind, p_2 objects of second kind,..., pn objects of nth kind, then the number of ways of choosing r objects out of these $(p_1 + p_2 + ...+ p_n)$ objects

= Coefficients of x^r in $(1 + x + ... + x^{p_1})(1 + x^2 + ... + x^{p_2})$

$$...(1 + x^2 + ... + x^{p_n}) \qquad\qquad ....(i)$$

This problem can also be stated as:

Let there be n distinct objects $x_1,...x_n... x_1$ can be used at the most p1 times, x_2 at the most p_2 times,..., x_n at the most p_n times. Then the number of ways to have r things is given by (i).

4. In how many ways can a pack of 52 cards be divided equally among four players in order?

Sol. Here 52 cards are to be divided into four equal groups and the order of the groups is important. So, the required number of ways.

$$\left[\frac{52!}{(13!)^4 \, 4!} \right]^{4!} = \frac{52!}{(13!)^4} .$$

Alternate method:
For the first player we have $^{52}C_{13}$ choices, for the second player $^{39}C_{13}$ choices, for the third player $^{26}C_{13}$. Choices and for the last player we have choices. Hence the total number of ways =

$$^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13} = \frac{52!}{(13!)^4} .$$

5. There are 6 letters and 6 directed envelopes. Find the number of ways in which all letters are put in the wrong envelopes.

Sol. Number of ways = $6!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} \right]$

$$= \frac{6!}{2!} - \frac{6!}{3!} + \frac{6!}{4!} - \frac{6!}{5!} + \frac{6!}{6!} = 360 - 120 + 30 - 6 + 1 = 265.$$

PROBABILITY THEORY

Some Standard Properties and Results

1. The set of all possible outcomes (results) in a trial is called **sample space** for the trial. It is denoted by S. The elements of S are called **sample points**.

2. Let S be a sample space of a random experiment. Every subset of S is called an **event**.

3. Two events A, B in a sample space S are said to be **disjoint** or **mutually exclusive** if $A \cap B = \varnothing$.

4. The events A_1, A_2, ..., A_n in a sample space S are said to be **mutually exclusive** or **pairwise disjoint** if every pair of the events A_1, A_2, ..., A_n are disjoint.

5. Two events A, B in a sample space S are said to be exhaustive if $A \cup B = S$.

6. The events A_1, A_2, ... A_n in a sample space S are said to be **exhaustive** if
$A_1 \cup A_2 \cup ... A_n = S$.

7. Two events A, B in a sample space S are said to be **complementary** if
$A \cup B = S, A \cap B = \varnothing$.

8. Let A be an event in a sample space S. An event B in a S is said to be **complement** of A if A, B are complementary in S. The complement B of A is denoted by $\overline{A}$.

9. Let S be a finite sample space and P be a probability function of S. If A is an event in S then P(A), the image of A, is called probability of A.

10. If A is an event in a sample space S, then $P\overline{A} = 1 - P(A)$.

11. Let A, B be two events in a sample space S. If $A \subseteq B$ then $P(A) \leq P(B)$.

12. Let S be a sample space containing n sample points. If E is an elementary event in S, then $P(E) = \dfrac{1}{n}$.

13. If A is an event in a sample space S, then the ratio $P(A) : P(\overline{A})$ is called the **odds favour** to A and $P(\overline{A}) : P(A)$ is called the **odds against** to A.

14. **Addition theorem on probability:** If A, B are two events in a sample space S, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

15. If A, B are two events in a sample space then the event of happening B after the event A happening is called **conditional event**. It is denoted by $B \mid A$.

16. If A, B are two events in a sample space S and $P(A) \neq 0$, then the probability of B after the event A has occurred is called **conditional probability** of B given A. It is denoted by $P(B \mid A)$.

17. If A, B are two events in a sample space S such that $P(A) \neq 0$, then $P(B \mid A) = \dfrac{n(A \cap B)}{n(A)}$.

18. Multiplication theorem of probability: Let A, B be two events in a sample space S such that $P(A) \neq 0$, $P(B) \neq 0$. Then
(i) $P(A \cap B) = P(A) P(B \mid A)$
(ii) $P(A \cap B) = P(B) P(A \mid B)$

19. Two events A, B in a sample space S are said to be independent if $P(B \mid A) = P(B)$.

20. Two events A, B in a sample space S are independent iff $P(A \cap B) = P(A) P(B)$.

21. If A_1, A_2 are two mutually exclusive and exhaustive events and E is any event then $P(E) = P(A_1) P(E \mid A_1) + P(A_2) P(E \mid A_2)$

22. If A_1, A_2, A_3 are two mutually exclusive and exhaustive events and E is any event then $P(E) = P(A_1) P(E \mid A_1) + P(A_2) P(E \mid A_2) + P(A_3) P(E \mid A_3)$.

Bayes' Theorem
If an event A can occur in combination with one of the mutually exclusive events B_1, B_2 ... B_n, then

$$P(B_k \mid A) = \frac{P(A \mid B_k) P(B_k)}{\displaystyle\sum_{i=1}^{n} P(A \mid B_i) P(B_i)}$$

Solved Examples

1. What is the probability of getting a multiple of 3 in a throw of a single die?

Sol. Total number of outcomes $n = 6$
Total number of favourable cases $m = 2$ (occurrence of 3 and 6)

Required probability $= \dfrac{m}{n} = \dfrac{2}{6} = \dfrac{1}{3}$

2. From a pack of 52 cards, one card is drawn at random. What is the probability that its either an ace or spade?

Sol. Let A be the event that the card drawn is an ace and B be the event that the card drawn is a spade. We know that, number of aces in a pack of 52 cards is 4.

i.e. $n(A) = 4 \Rightarrow P(A) = \dfrac{4}{52} = \dfrac{1}{13}$

Also $n(B) = B \Rightarrow P(B) = \dfrac{13}{52} = \dfrac{1}{4}$

Also, there is one card which is both an ace and a spade.

i.e. $n(A \cap B) = 1 \Rightarrow P(A \cap B) = \dfrac{1}{52} = \dfrac{4}{52} + \dfrac{13}{52} - \dfrac{1}{52} = \dfrac{16}{52} = \dfrac{4}{13}$

$\therefore$ The probability of occurrence of either an ace or a spade is $\dfrac{4}{13}$.

3. In a bolt factory — machines A, B and C manufacture respectively 25, 35 and 40% of the total. Out of their output 5, 4 and 2% are defective bolts. A bolt is drawn from the produce and is found defective. What are the probabilities that it was manufactured by A, B and C?

Sol. Define events E, A_1, A_2, A_3 as
E: The bolt is defective
A_1 = The bolt is produced by machine A
A_2 = The bolt is produced by machine B
A_3 = The bolt is produced by machine C
Then, $P(A_1) = 0.25$, $P(A_2) = 0.35$ and $P(A_3) = 0.4$

$P\left(\dfrac{E}{A_1}\right) = 0.05$, $P\left(\dfrac{E}{A_2}\right) = 0.04$, $P\left(\dfrac{E}{A_3}\right) = 0.02$

We are supposed to find $P\left(\dfrac{A_1}{E}\right)$, $P\left(\dfrac{A_2}{E}\right)$ and $P\left(\dfrac{A_3}{E}\right)$

Now, using Bayes' theorem,

$$P\left(\frac{A_1}{E}\right) = \frac{P(A_1).P\left(\dfrac{E}{A_1}\right)}{P(A_1).P\left(\dfrac{E}{A_1}\right) + P(A_2).P\left(\dfrac{E}{A_2}\right) + P(A_3).P\left(\dfrac{E}{A_3}\right)}$$

$$= \frac{0.25 \times 0.05}{(0.25 \times 0.05) + (0.35 \times 0.04) + (0.4 \times 0.02)} = \frac{25}{69}$$

Similarly, $P\left(\dfrac{A_2}{E}\right) = \dfrac{28}{69}$

and $P\left(\dfrac{A_3}{E}\right) = 1 - P\left(\dfrac{A_1}{E}\right) - P\left(\dfrac{A_2}{E}\right) = 1 - \dfrac{25}{69} - \dfrac{28}{69} = \dfrac{16}{69}$

4. If the probability that A fails in the examination is 0.25 and probability that B fails in the examination is 0.5, then the probability that either A or B fails in the examination is

  a. $\dfrac{1}{8}$ b. $\dfrac{3}{8}$ c. $\dfrac{5}{8}$ d. $\dfrac{7}{8}$

Sol. Let us define two events as

A: A fails, B: B fails, then we have to find probability of A or B i.e.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$, As failing of A is independent of failing of B thus we have

$$P(A \cup B) = P(A) + P(B) - P(A).P(B) = \frac{1}{4} + \frac{1}{2} - \frac{1}{4}.\frac{1}{2} = \frac{2+4-1}{8} = \frac{5}{8}$$

12 Statistics

Median

The median is that value of the variable which divides the group in to two equal parts. One part comprises all the values greater than and the other part comprises all the values less than the median.

Calculation of Median
For individual observations

Step 1   Arrange the observations x_1, x_2, ..., x_n in ascending or descending order of magnitude.

Step 2   Determine the total number of observations, say, n

Step 3   If n is odd, then median is the value of $\left(\dfrac{n+1}{2}\right)^{th}$ observation. If n is

even, then median is the AM of the values of $\left(\dfrac{n}{2}\right)^{th}$ and $\left(\dfrac{n}{2}+1\right)^{th}$

observations.

For discrete frequency distribution.

Step 1   Find the cumulative frequencies (c.f.)

Step 2   Find $\dfrac{N}{2}$, where $N = \sum\limits_{i=1}^{n} f_i$

Step 3   See the cumulative frequency (c.f.) just greater than $\dfrac{N}{2}$ and determine

the corresponding value of the variable, which is the median.

For grouped or continuous frequency distribution.

Step 1   Obtain the frequency distribution.

Step 2   Prepare the cumulative frequency column and obtain $N = \Sigma f_i$ Find

$\dfrac{N}{2}$.

Step 3 See the cumulative frequency just greater than $\frac{N}{2}$ and determine the corresponding class. This class is known as the median class.

Step 4 Use the formula: Median $= L + \left(\dfrac{N/2 - F}{f} \right) \times h$, where, $L =$ lower limit of the median class, f is frequency of the median class, $h =$ width (size) of the median class, $F =$ cumulative frequency of the class preceding the median class, $N = \Sigma f_i$.

Note: The **mean deviation** from the median for any distribution is minimum.

Quartiles

The values which divide the given data into four equal parts are known as quartiles. Obviously, there will be three such points Q_1, Q_2 and Q_3 (such that $Q_1 \le Q_2 \le Q_3$) termed as three quartiles. Mathematically,

$$Q_1 = 1 + \frac{h}{f}\left(\frac{N}{4} - C \right),\ Q_2 = \text{Median},\ \ Q_3 = 1 + \frac{h}{f}\left(\frac{3N}{4} - C \right)$$

where $l =$ Lower limit of the class containing Q_1 or Q_3
$f =$ Frequency of the class containing Q_1 or Q_3
$h =$ Magnitude of the class containing Q_1 or Q_3
$C =$ Cumulative frequency of class preceding the class containing Q_1 or Q_3

Mode

Mode is the value which occurs most frequently in a set of observations. The mode may or may not exist, and even if it does exist, it may not be unique. A distribution having a unique mode is called 'unimodal' and one having more than one is called 'multimodal'.

Examples

The set 2, 2, 5, 7, 9, 9, 9, 10, 10, 11, 12, 18 has mode 9.
The set 3, 5, 8, 10, 12, 15, 16 has no mode.
The set 2, 3, 4, 4, 4, 5, 5, 7, 7, 7, 9 has two modes, 4 and 7, and is called bimodal.

Calculation of Mode

In case of frequency distribution, mode is the value of the variable corresponding to the maximum frequency. In case of continuous frequency distribution, the class corresponding to the maximum frequency is called the modal class and the value of mode is obtained as, Mode =

$$I + \frac{h(f_1 - f_0)}{(f_1 - f_0) - (f_2 - f_1)}\ ,\ \text{where}$$

$\ell =$ Lower limit of modal class
$h =$ Magnitude of the modal class, $f_1 =$ frequency of the modal class
$f_0 =$ Frequency of class preceding the modal class
$f_2 =$ Frequency of class succeeding the modal class

The above formula can be rephrased as $\text{Mode} = L_1 + \left(\dfrac{D_1}{D_1 + D_2}\right) c$,

where L_1 = Lower class boundary of the modal class (i.e. the class containing the mode)
D_1 = Excess of modal frequency over frequency of next lower class
D_2 = Excess of modal frequency over frequency of next higher class
c = Size of the model class interval.

Relationship Between Mean (M), Median (Md) and Mode(M_0)
In case of symmetrical distribution, Mean = Median = Mode
In case of a '**moderately**' asymmetrical distribution, **Mode = 3 Median – 2 Mean**

Measures of Dispersion
The various measures of dispersion are as follows
1. Range **2.** Mean deviation
3. Standard deviation **4.** Quartile deviation
5. 10 – 90 percentile range

Range
It is the difference between two extreme observations of a distribution. Let X_{max} be the greatest observation and X_{min} the smallest observation of the variable. Then, $\text{Range} = X_{max} - X_{min}$.

Mean Deviation (MD)
If $X_1, X_2, X_3 ... X_n$ are n given observations, then the mean deviation (MD) about A, is given by

$$MD = \frac{1}{n} \Sigma |X_i - A| = \frac{1}{n} \Sigma |d_i|, \text{ where } d_i = X_i - A.$$ Modulus removes the effects of negative deviations and therefore gives us the absolute value of the deviation.
In case of frequency distribution, mean deviation about A is given by

$$\frac{1}{n} \Sigma f_i |X_i - A| = \frac{1}{n} \Sigma f_i |d_i| = \overline{|X - \bar{X}|}$$

Standard Deviation
If $X_1, X_2, ... X_N$ is the set of N observations, then its standard deviation is given

by $\sigma = \sqrt{\dfrac{1}{N} \Sigma (X_i - \bar{X})^2}$ where $\bar{X}$ is the AM alternative formula for

standard deviations is $\sigma = \sqrt{\dfrac{\Sigma f_i X_i^2}{N} - \left(\dfrac{\Sigma f_i X_i}{N}\right)^2} = \sqrt{\overline{X^2} - \bar{X}^2}$

The root mean square deviation about any point a is defined as S =

$$\sqrt{\dfrac{\sum\limits_{i=1}^{N}(X_i - a)^2}{N}}$$, where a is the average

Variance

It is the square of the standard deviation.

Therefore, variance = $\sigma^2 = \dfrac{1}{N} \Sigma \ (X_i - \bar{X})^2$. Generally, s^2 represents

sample variance and σ^2 represents population variance. Sample variance means variance of a sample drawn out from a population.

Quartile Deviations. (QD) or Semi Inter-Quartile Range

Inter quartile range = $Q_3 - Q_1$;   Quartile deviation (QD) = $\dfrac{Q_3 - Q_1}{2}$

(or Semi inter-quartile range) Quartile deviation is more commonly used as a measure of dispersion than inter-quartile range.

For a symmetrical distribution, $Q_3 - m_d = m_d - Q_1 \Rightarrow m_d = \dfrac{Q_3 + Q_1}{2}$

Coefficient of Variation : CV = $\dfrac{\sigma}{X} \times 100$

Note that the coefficient of variation is independent of the units used. For this reason, it is useful in comparing distribution where the units may be different. A disadvantage of the coefficient of variation is that it fails to be used when $\bar{X}$ is close to zero.

Linear Programming Problem (LPP)

Given a set of m linear inequalities or equations in n variables, we wish to find non-negative values of these variables which will satisfy these inequalities. The inequalities or equations are called the **constraints** and the function to be maximized or minimized is called the **objective function**, which can be of maximization type or minimization type. The general form of linear programming problem is maximize (or minimize),

$z = c_1x_1 + c_2x_2 + ... + c_nx_n$ ... (i)

Subject to

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n} . x_n\{\le, =, \ge\} b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n} . x_n\{\le, =, \ge\} b_2$ ... (ii)

$a_{m1}x_1 + a_{m2} x_2 + ... + a_{mn} . x_n\{\le, =, \ge\} b_m$

and $x_1, x_2, x_3, ..., x_n \ge 0$ ... (iii)

where

(i) $x_1, x_2, ..., x_n$ are the variables whose values we wish to determine and are called the **decision variables**,

(ii) the linear function z which is to be maximised or minimised is called the **objective function**,

(iii) the inequalities or equations in (ii) are called the **constraints**,

(iv) the set of inequalities in (iii) is known as the set of **non-negativity restrictions**,

(v) the expression ($\leq = \geq$) means that one and only one of the signs $\leq = \geq$ holds for a particular constraint but the sign may vary from constraint to constraint.

Solution
A set of values of the decision variables which satisfy the constraints of a linear programming problem (LPP) is called a **solution of the LPP**.

Feasible solution
A solution of an LPP which also satisfies the non-negativity restrictions of the problem is called its **feasible solution**. The set of all feasible solutions of an LPP is called the **feasible region**.

Solved Examples

1. Calculate the median for the following distribution:

Class:	5-10	10-15	15-20	20-25	25-30	30-35	35-40	40-45
Frequency:	5	6	15	10	5	4	2	2

Sol. Calculation of median

Class	Frequency	Cumulative frequency
5-10	5	5
10-15	6	11
15-20	15	26
20-25	10	36
25-30	5	41
30-35	4	45
35-40	2	47
40-45	2	49
	N = 49	

Here $N = 49 \Rightarrow \dfrac{N}{2} = \dfrac{49}{2} = 24.5$. The cumulative frequency just greater than

$\dfrac{N}{2}$ is 26 and the corresponding class is 15-20. Thus 15-20 is the median class such that

i = 15, f = 15, F = 11, h = 5.

$$\therefore \text{Median} = 1 + \frac{\dfrac{N}{2} - F}{f} \times h = 15 + \frac{24.5 - 11}{15} \times 5 = 15 + \frac{13.5}{15} \times 5 = 19.5$$

2. Table below shows a frequency distribution of the weekly wages of 65 employees. Find the quartiles Q_1, Q_2 and Q_3

Wages	Number of employees
250.00 - 259.99	8
260.00 - 269.99	10
270.00 - 279.99	16
280.00 - 289.99	14
290.00 - 299.99	10
300.00 - 309.99	5
310.00 - 319.99	2
Total	**65**

Sol. The first quartile Q_1 is wage obtained by counting $\dfrac{N}{4} = \dfrac{65}{4} = 16.25$ of the cases, beginning with the first (lowest) class. Since the first class contains 8 cases, we must take 8.25 (from 16.25 – 8) of the 10 cases from the second class. Using the method of linear interpolation, we have

$$Q_1 = 259.995 + \frac{8.25}{10}\,(10) = \text{Rs. } 268.25.$$

$$Q_2 = 269.995 + \frac{14.5}{16}\,(10) = \text{Rs. } 279.06.$$

$$Q_3 = 289.995 + \frac{0.75}{10}\,(10) = \text{Rs. } 290.75.$$

3. Find the mean, median, and mode for the sets
(a) 3, 5, 2, 6, 5, 9, 5, 2, 8, 6 and
(b) 51.6, 48.7, 50.3, 49.5, 48.9

Sol. (a) Arranged in an array, the numbers are 2, 2, 3, 5, 5, 5, 6, 6, 8 and 9.
Mean = 5.1; Median = Arithmetic mean of two middle numbers

$$= \frac{1}{2}\,(5 + 5) = 5;$$

Mode = Number most frequently occurring = 5.
(b) Arranged in an array, the numbers are 48.7, 48.9, 49.5, 50.3 and 51.6.
Mean = 49.8; Median = middle number = 49.5; Mode = non existent.

4. Compute the mode of the following distribution:

Class intervals:	0-7	7-14	14-21	21-28	28-35	35-42	42-49
Frequency :	19	25	36	72	51	43	28

Sol. Mode $(M_0) = 1 + \dfrac{f - f_{-1}}{2f - f_{-1} - f_1} \times i = 21 + \dfrac{72 - 36}{144 - 36 - 51} \times 7 = 21 + \dfrac{252}{57}$.

$= 21 + 4.42 = 25.42$

5. Find the mean wage from the data given below.

Wage (Amount in Rs.)	800	820	860	900	920	980	1000
Number of workers	7	14	19	25	20	10	5

Sol. Let the assumed mean be $A = 900$, $h = 20$

x_i	f_i	$x_i - A$	$u_i = (x_i - A)/h$	$f_i u_i$
800	7	-100	-5	-35
820	14	-80	-4	-56
860	19	-40	-2	-38
900	25	0	0	0
920	20	20	1	20
980	10	80	4	40
1000	5	100	5	25
	$\Sigma f_i = 100$			$\Sigma f_i u_i = -44$

Here $A = 900$, $h = 20$.

$$\therefore \text{Mean} = \overline{X} = A + h\left(\frac{1}{N}\sum_{i=1}^{n} f_i \ u_i\right) = 900 + 20\left(-\frac{44}{100}\right) = 891.2$$

Hence, mean wage = Rs. 891.2.

With the help of this method, calculation of multiplying two inconvenient numbers is avoided.

6. (a) Prove that $\sigma = \sqrt{\dfrac{\Sigma X^2}{N} - \left(\dfrac{\Sigma X}{N}\right)^2} = \sqrt{\overline{X^2} - \overline{X}^2}$

(b) Use the formula in part (a) to find the standard deviation of the set 12, 6, 7, 3, 15, 10, 18, 5.

Sol. (a) By definition, $\sigma = \sqrt{\dfrac{\sum (X - \overline{X})^2}{N}}$. Therefore,

$$\sigma^2 = \frac{\sum (X - \overline{X})^2}{N} = \frac{\sum (X^2 - 2X\overline{X} + \overline{X}^2)}{N} = \frac{\sum X^2 - 2\overline{X}\sum X + N\overline{X}^2}{N}$$

$$\frac{\sum X^2}{N} - 2\overline{X}\frac{\sum X}{N} + \overline{X}^2 = \overline{X^2} - 2\overline{X}^2 + \overline{X}^2 = \overline{X^2} - \overline{X}^2 = \frac{\sum X^2}{N} - \left(\frac{\sum X}{N}\right)^2$$

(b) $\overline{X^2} = \dfrac{\sum X^2}{N} = 114$ and $\overline{X} = \dfrac{\sum X}{N} = 9.5$.

Therefore, $\sigma = \sqrt{114 - 90.25} = 4.87$.

7. A manufacturer can produce two different products A and B during a given time period. Each of these products requires four different manufacturing operations: grinding, turning, assembly and testing. The manufacturing requirements in hours per unit of product are given below for A and B.

	A	B
Grinding	1	2
Turning	3	1
Assembly	6	3
Testing	5	4

The available capacities of these operations in hours for given time period are: grinding 30; turning 60; assembly 200; testing 200. The contribution to profit is Rs. 2 for each unit of A and Rs. 3 for each unit of B. The firm can sell all that it produce at the prevailing market price. Now answer the following questions.

i. The key decision to be made is to determine number of
   a. hours required for grinding and turning products A and B
   b. hours required for assembling and testing products A and B
   c. Both (a) and (b)
   d. units of products A and B which will optimize the profits

ii. The objective is to
   a. minimize the total hours required for grinding, turning, assembly and testing
   b. minimize the total cost of production
   c. maximize the total profit
   d. minimize the profit
   If x_1 and x_2 are the numbers of units of products A and B respectively, which the company decided to produce, then

iii. The objective function (z) is
   a. $2x_1 + 2x_2$ b. $2x_1 + 3x_2$ c. $3x_1 + 2x_2$ d. $x_1 + 2x_2$

iv. The constraint for grinding is
   a. $x_1 + 2x_2 \leq 30$ b. $3x_1 + x_2 \geq 60$
   c. $x_1 + 2x_2 = 30$ d. $x_1 + 2x_2 > 30$

v. The constraint for turning is
  a. $3x_1 + x_2 > 60$
  b. $3x_1 + x_2 = 60$
  c. $3x_1 + x_2 = 0$
  d. $3x_1 + x_2 \leq 60$

vi. The constraint for assembly is
  a. $6x_1 + 3x_2 = 200$
  b. $6x_1 + 3x_2 > 200$
  c. $6x_1 + 6x_2 = 0$
  d. $6x_1 + 3x_2 \leq 200$

vii. The constraint for testing is
  a. $5x_1 + 3x_2 \leq 200$
  b. $5x_1 + 3x_2 \geq 0$
  c. $5x_1 + 4x_2 \geq 200$
  d. $5x_1 + 4x_2 \leq 200$

viii. The non-negative restrictions are
  a. $x_1 \leq 0, x_2 \geq 0$
  b. $x_1 \geq 0, x_2 \geq 0$
  c. $x_2 \leq 0 \leq x_1$
  d. $x_1 \leq 0, x_2 \leq 0$

Sol.
  i. The key decision to be made is to determine the number of units of products A and B to be produced by the company. Hence, the correct answer is (d).
  ii. The objective is to maximize the total profit. Hence, the correct answer is (c).
  iii. The objective function (z), i.e. the total profit the manufacturer gets after selling the two products A and B, is given by $Z = 2x_1 + 3x_2$. Hence, the correct answer is (b).
  iv. In order to produce these two products A and B, the total number of hours required at the grinding centre are $x_1 + 2x_2$. Since the manufacturer does not have more that 30 hr available in grinding centre, $x_1 + 2x_2 \leq 30$. Hence, the correct answer is (a).
  v. The total number of hours required at the turning centre is $3x_1 + x_2$. Since the manufacturer does not have more than 60 hr available in turning centre, $3x_1 + x_2 \leq 60$. Hence, the correct answer is (d).
  vi. The total number of hours required at the assembly centre is $6x_1 + 3x_2$. Since the manufacturer does not have more than 200 hr in the assembly centre, $6x_1 + 3x_2 \leq 200$. Hence, the correct answer is (d).
  vii. The total number of hours required at the testing centre is $5x_1 + 4x_2$. Since the manufacturer does not have more than 200 hr in the testing centre, we must have $5x_1 + 4x_2 \leq 200$. Hence, the correct answer is (d).
  viii Since it is not possible for the manufacturer to produce negative number of the products, we must also have $x_1 \geq 0, x_2 \geq 0$. Hence, the correct answer is (b).

Practice Exercise

1. If $Z^3 + 3iZ^2 - 3Z - i = 0$, then find the value of $Z^4 + Z^3 + Z^2 + Z +$.
    a. 1   b. -1 c. i   d. $-i$

2. The modulus of $e^{a + ib}$ is
    a. e^a b. e^{a+b} c. e^{a-b} d. $e^a \cdot b$

3. The complex number Z is represented by a point P on the line $3x + 4y = 12$. The least value of $|Z|$ is

    a. 12   b. 0   c. $\dfrac{24}{5}$ d. $\dfrac{12}{5}$

4. Find the number of solutions of the equation $|Z - i| = |Z + i|$
    a. 2   b. 1
    c. 4   d. Infinitely many

5. Given $\left(\dfrac{2 + \sqrt{3} + i}{2 + \sqrt{3} - i} \right)^{12}$ $= a + ib$. Then $a + b$ will be equal to

    a. 2   b. zero   c. 3   d. 1

6. $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = k \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$,

    then find the value of K.
    a. 1   b. 2   c. 3   d. 4

7. The value of $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix}$ is

    a. -2 b. 2   c. $x^2 + 2$ d. None of these

8. The inverse of the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ is

    a. $\begin{bmatrix} \dfrac{3}{7} & \dfrac{1}{7} \\ \dfrac{1}{7} & \dfrac{2}{7} \end{bmatrix}$ b. $\begin{bmatrix} -\dfrac{3}{7} & -\dfrac{1}{7} \\ -\dfrac{1}{7} & -\dfrac{2}{7} \end{bmatrix}$ c. $\begin{bmatrix} \dfrac{3}{7} & \dfrac{1}{7} \\ -\dfrac{1}{7} & \dfrac{2}{7} \end{bmatrix}$ d. $\begin{bmatrix} \dfrac{3}{7} & -\dfrac{1}{7} \\ \dfrac{1}{7} & \dfrac{2}{7} \end{bmatrix}$

9. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then the value of adjA. is

    a. $\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ b. $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ d. Does not exist

10. If the matrix $\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$ is expressible as $(P + Q)$, where P is symmetric and Q is skew-symmetric, then P =

    a. $\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$ b. $\begin{pmatrix} 3 & -2 \\ -2 & -1 \end{pmatrix}$ c. $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ d. $\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$

11. Find the coefficient of x^7 in the expansion of $(x - 2x^2)^{-3}$.
    a. 67584   b. 72424   c. 56842   d. None of these

12. Find the coefficient of x^n in the expansion of $\dfrac{e^{7x} + e^x}{e^{3x}}$.

    a. $\dfrac{2^n}{n!}(-1)^n$ b. $\dfrac{2^n}{n!}(2^n + (-1^n)$

    c. $\dfrac{2^n}{n!}(2^n - (-1)^n)$ d. None of these

13. Sum the series: $\dfrac{1}{1!} + \dfrac{1+3}{2!}x + \dfrac{1+3+5}{3!}x^2 + \dfrac{1+3+5+7}{4!}x^3 + ...$
    a. $(x + 2)e^x$ b. $(x - 1)e^x$ c. $(x + 1)e^x$ d. $(x - 2)e^x$

14. Sum the following series to infinity.

$$1 + \frac{1+a}{2!} + \frac{1+a+a^2}{3!} + \frac{1+a+a^2+a^3}{4!} + ...$$

    a. $\dfrac{e^a - e}{2}$ b. $\dfrac{e^a + e}{a - 1}$ c. $\dfrac{e^a - 1}{a - 1}$ d. $\dfrac{e^a - e}{a - 1}$

15. Find the sum $\dfrac{3^2}{1!} + \dfrac{5^2}{3!} + \dfrac{7^2}{5!} + ..., \infty$.
    a. 5e   b. 4e   c. 6e   d. None of these

16. $\lim\limits_{x \to \infty} \left(\dfrac{x+1}{x+2}\right)^{2x+1} =$

    a. e   b. e^{-2} c. e^{-1} d. 1

17. $\lim\limits_{x \to 0} \dfrac{(\cos x)^{\frac{1}{2}} - (\cos x)^{\frac{1}{3}}}{\sin^2 x} =$

    a. $\dfrac{1}{6}$ b. $-\dfrac{1}{12}$ c. $\dfrac{2}{3}$ d. $\dfrac{1}{3}$

18. If $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$, then find the value of

    $\lim\limits_{h \to 0} \dfrac{f(1-h) - f(1)}{h^3 + 3h}$ is

    a. $\dfrac{50}{3}$ b. $\dfrac{22}{3}$ c. 13   d. None of these

19. Find the value of constant a, b such that

    $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + 1}{x + 1} - ax - b\right) = 0$

    a. $a = b = 0$ b. $a = 1, b = -1$ c. $a = b = 1$ d. $a = 2, b = -1$

20. If $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 2, & x \text{ is irrational} \end{cases}$, then

    a. f(x) is continuous in R ~ {1}
    b. f(x) is continuous in R ~ Q
    c. f(x) is continuous in R but not differentiable in R
    d. f(x) is neither continuous nor differentiable in R

21. Let $f(x + y) = f(x)\, f(y)$ for all x and y. If $f(5) = 2$ and $f'(0) = 3$, then $f'(5)$ is equal to

    a. 5   b. 6   c. 0   d. 3

22. If $a(x^4 - y^4) = \sqrt{1 - x^8} + \sqrt{1 - y^8}$ then $\dfrac{dy}{dx} =$

    a. $\dfrac{y}{x}$ b. $\dfrac{x^3}{y^3}\sqrt{\dfrac{1 - y^8}{1 - x^8}}$

    c. $\dfrac{ax}{y}\sqrt{\dfrac{1 - x^8}{1 - y^8}}$ d. None of these

23. If $\tan (x + y) + \tan (x - y) = 1$, then $\dfrac{dy}{dx} =$

a. $\dfrac{(x + y)\ \sec^2(x + y)}{(x - y)\ \sec^2(x - y)}$

b. $\dfrac{\sec^2(x + y) - \sec^2(x - y)}{(x + y)\sec^2(x + y) + \sec^2(x - y)}$

c. $\dfrac{x - y}{(x + y)} \cdot \dfrac{\sec^2(x + y)\ + \sec^2(x - y)}{\sec^2(x + y) - \sec^2(x - y)}$

d. $-\dfrac{\sec^2(x + y) + \sec^2(x - y)}{\sec^2(x + y) - \sec^2(x - y)}$

24. If $y = \tan^{-1}\left(\dfrac{\sqrt{1 + x^2} - 1}{x}\right)$, then $y'(0) =$

a. $\dfrac{1}{2}$ b. 0   c. 1   d. Does not exist

25. If $y = ae^x + be^{2x} + ce^{3x}$, which of the following is true?
a. $y_3 - 6y_2 + 11y_1 - 6y = 0$ b. $y_2 - 11y_1 + 6y = 0$
c. $y_3 + 6y_2 - 11y_1 + 6y = 0$ d. None of these

26. $\displaystyle\int (7x - 2)\ \sqrt{3x + 2}\ dx$

a. $\dfrac{14}{15}(3x + 2)^{\frac{5}{2}} - \dfrac{40}{9}(3x + 2)^{\frac{3}{2}} + c$

b. $\dfrac{14}{45}(3x + 2)^{\frac{5}{2}} - \dfrac{40}{27}(3x + 2)^{\frac{3}{2}} + c$

c. $\dfrac{14}{15}(3x + 2)^{\frac{3}{2}} - \dfrac{40}{27}(3x + 2)^{\frac{3}{2}} + c$

d. $\dfrac{7}{14}(3x + 2)^{\frac{5}{2}} - \dfrac{20}{9}(3x + 2)^{\frac{3}{2}} + c$

27. $\int \dfrac{e^x}{e^{2x} + 6e^x + 5} \, dx$

a. $\dfrac{1}{4} \tan^{-1} \dfrac{e^x + 1}{4} + c$

b. $\dfrac{1}{4} \log \left| \dfrac{e^x + 1}{e^x + 5} \right| + c$

c. $\dfrac{1}{4} \log \left| \dfrac{e^x - 1}{e^x + 1} \right| + c$

d. None of these

28. If $\int \dfrac{dx}{5 + 4\cos x} = K \tan^{-1} \left(M \tan \dfrac{x}{2} \right) + C$, then which of the following is true?

a. $K = 1$

b. $K = \dfrac{2}{3}$

c. $M = \dfrac{1}{4}$

d. $M = \dfrac{2}{3}$

29. The value of the integral $\int_2^4 \dfrac{\sqrt{x^2 - 4}}{x^4} \, dx$ is

a. $\sqrt{\dfrac{3}{32}}$

b. $\dfrac{\sqrt{3}}{32}$

c. $\dfrac{32}{\sqrt{3}}$

d. $-\dfrac{\sqrt{3}}{32}$

30. The value of the integral $\displaystyle\int_0^{\pi/2} \dfrac{d\theta}{4\cos^2\theta + 9\sin^2\theta}$

a. $\dfrac{\pi}{4}$

b. $\dfrac{\pi}{6}$

c. $\dfrac{\pi}{12}$

d. None of these

31. The coordinates of a point are (0, 1) and the ordinate of another point is –3. If the distance between the two points is 5, then the abscissa of another point is

a. 3

b. –3

c. ± 3

d. 1

32. Find the coordinates of the point which divides, internally and externally, the line joining (–1, 2) to (4, –5) in the ratio 2 : 8.

a. $\left(0, \dfrac{3}{5} \right), \left(-\dfrac{8}{3}, \dfrac{13}{3} \right)$

b. $\left(\dfrac{8}{3}, -\dfrac{13}{3} \right), \left(0, \dfrac{5}{3} \right)$

c. $\left(-\dfrac{8}{3}, \dfrac{13}{3} \right), \left(0, \dfrac{3}{5} \right)$

d. None of these

33. Find the value of x_1 if the distance between the points $(x_1, 2)$ and $(3, 4)$ be 8.

    a. $3 \pm 2\sqrt{15}$ b. $3 + 2\sqrt{15}$ c. $3 - 2\sqrt{5}$ d. None of these

34. If $A(2, 2)$, $B(-4, -4)$ and $C(5, -8)$ are the vertices of any triangle, then the length of median passing through C will be

    a. $\sqrt{65}$ b. $\sqrt{117}$ c. $\sqrt{85}$ d. $\sqrt{113}$

35. If a, b, c are in harmonic progression, then straight line $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{1}{c} = 0$ always passes through a fixed point, which is

    a. $(-1, -2)$ b. $(-1, 2)$ c. $(1, -2)$ d. $(1, -\dfrac{1}{2})$

36. The vectors $\vec{a}$ and $\vec{b}$ are non-collinear. The value of p for which the vectors $\vec{c} = (p+1)\vec{a} + 2\vec{b}$ and $\vec{d} = (2p-1)\vec{a} - \vec{b}$ are collinear is

    a. $\dfrac{1}{3}$ b. $\dfrac{1}{5}$ c. -3 d. -5

37. If the vectors $\hat{i} + \lambda\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar, then the value of λ is

    a. $-\dfrac{1}{2}$ b. $-\dfrac{1}{3}$ c. 3 d. -2

38. If the vectors $2\hat{i} - \hat{j} - \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + a\hat{j} + 5\hat{k}$ are coplanar, then the value of a is

    a. 1   b. 2   c. -4 d. 4

39. Let S, T and U be the middle points of the sides QR, RP, PQ respectively of a ΔPQR. Then $\vec{PS} + \vec{QT} + \vec{RU}$

    a. 0   b. 2   c. $\vec{O}$ d. None of these

40. If non-zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other, then the solution of the equation $\vec{r} \times \vec{a} = \vec{b}$ is given by

    a. $\vec{r} = x\vec{a} + \dfrac{\vec{a} \times \vec{b}}{|\vec{a}|^2}$ b. $\vec{r} = x\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$

    c. $\vec{r} = x(\vec{a} \times \vec{b})$ d. $\vec{r} = x(\vec{b} \times \vec{a})$

41. If the position vectors of three points P, Q and R are respectively $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$ and $7\hat{i}+4\hat{j}+9\hat{k}$, then the unit vector perpendicular to the plane of the $\triangle PQR$ is

a. $\dfrac{31\hat{i}+28\hat{j}+9\hat{k}}{\sqrt{2405}}$

b. $\dfrac{31\hat{i}-28\hat{j}-9\hat{k}}{\sqrt{2405}}$

c. $\dfrac{31\hat{i}-38\hat{j}-9\hat{k}}{\sqrt{2486}}$

d. None of these

42. Match the following two lists given that $A + B + C = \pi$

(A) sin 2A + sin 2B + sin 2C   I. $1+4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(B) sin A + sin B + sin C   II. tan A tan B tan C

(C) cos A + cos B + cos C   III. 4 sin A sin B sin C

(D) tan A + tan B + tan C   IV. $4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

a. A-IV, B-II, C-I, D-III   b. A-III, B-IV, C-I, D-II
c. A-I, B-III, C-IV, D-II   d. A-III, B-I, C-IV, D-II

43. The value of sin 12° sin 48° sin 54° is

a. $\dfrac{1}{4}$ b. $\dfrac{1}{8}$ c. $\dfrac{1}{16}$ d. None of these

44. If $\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}$, then the most general value of θ is

a. $n\pi+(-1)\dfrac{\pi}{4}$

b. $n\pi+(-1)\dfrac{\pi}{4}-\dfrac{\pi}{3}$

c. $n\pi+\dfrac{\pi}{4}-\dfrac{\pi}{3}$

d. $n\pi+(-1)\dfrac{\pi}{4}-\dfrac{\pi}{3}$

45. If 3 sin x + 4 cos x = 5, then find the value of $\tan\dfrac{x}{2}$

a. $\dfrac{1}{6}$ b. $\dfrac{1}{\sqrt{6}}$ c. $\dfrac{1}{3}$ d. $\dfrac{1}{\sqrt{3}}$

46. If $\tan\theta=-\dfrac{1}{\sqrt{3}}$ and $\sin\theta=\dfrac{1}{2}$, $\cos\theta=-\dfrac{\sqrt{3}}{2}$, then the principal value of θ will be

a. $\dfrac{\pi}{6}$ b. $\dfrac{5\pi}{6}$ c. $\dfrac{7\pi}{6}$ d. $-\dfrac{\pi}{6}$

47. Which of the following can be a possible value of $\sin^{-1}\dfrac{3}{5} + \cos^{-1}\dfrac{12}{13}$

a. $\sin^{-1}\dfrac{65}{56}$ b. $\sin^{-1}\dfrac{56}{65}$ c. $\sin^{-1}\dfrac{13}{25}$ d. None of these

48. If $\cot^{-1} x + \tan^{-1} 3 = \dfrac{\pi}{2}$, then find the value of x

a. $\dfrac{1}{3}$ b. $\dfrac{1}{4}$ c. 3   d. None of these

49. If a flag staff of 6 m high placed on the top of a tower throws a shadow of $2\sqrt{3}$ m along the ground, then the angle (in degrees) that the sun makes with the ground is

a. 60°   b. 30°   c. 45°   d. None of these

50. A committee of 5 is to be formed out of 6 gentlemen and 4 ladies. In how many ways this can be done, when (i) at least two ladies are included, (ii) at most two ladies are included?

a. 186   b. 164   c. 242   d. None of these

51. The principal wants to arrange 5 students on the platform such that the boy 'SALIM' occupies the second position and the girl 'SITA' is always adjacent to the girl 'RITA'. How many such arrangements are possible?

a. 12   b. 16   c. 8   d. 24

52. How many five-lettered words containing 3 vowels and 2 consonants can be formed using the letters of the word 'EQUATION' so that the two consonants occur together?

a. 1684   b. 980   c. 1264   d. 1440

53. In the simultaneous tossing of two perfect coins, the probability of having at least one head is

a. $\dfrac{1}{2}$ b. $\dfrac{1}{4}$ c. $\dfrac{3}{4}$ d. 1

54. The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then find $P(\overline{A}) + P(\overline{B})$.

a. 1.4   b. 1.2   c. 1.6   d. 1.8

55. Amit can solve 90% of the problems given in a book and Bhim can solve 70%. What is the probability that at least one of them will solve the problem, selected at random from the book?

a. 0.98   b. 0.95   c. 0.96   d. 0.97

56. Compute the median for the following data:

Mid-value : 115   125   135   145   155   165   175   185   195
Frequency :   6   25   48   72   116   60   38   22   3

a. 153.8   b. 155.6   c. 162.4   d. None of these

57. If the median of the following frequency distribution is 46, find the sum of the missing frequencies:

Variable: 10-20   20-30   30-40   40-50   50-60   60-70   70-80   Total
Frequency : 12   30   ?   65   ?   25   18   229

a. 76   b. 79   c. 74   d. 82

58. Find the mean deviation of the heights of the 100 male students.

Height (inches)	Number of students
60-62	5
63-65	18
66-68	42
69-71	27
72-74	8
	Total = 100

a. 2.94 inch   b. 2.26 inch   c. 2.76 inch   d. 2.48 inch

59. Find the standard deviation of the heights of the 100 male students as given in above question 10.
a. 2.64 inch   b. 2.48 inch   c. 2.84   d. 2.92 inch

60. The objective of a diet problem is to ascertain the quantities of certain foods that should be eaten to meet certain nutritional requirements and minimize cost. The consideration is limited to milk, beef and eggs and to vitamins A, B, C. The number of milligrams of each of these vitamins contained with a unit of each food is given below:

Vitamin	Litre of milk	kg of beef	Dozen eggs	Minimum daily requirement
A	1	1	10	1 mg
B	100	10	10	50 mg
Vitamin	10	100	10	10 mg
Cost	Rs. 15	Rs. 75	Rs. 2	

Formulate the problem as a linear programming problem.

Answer Key and Explanations

Higher Mathematics Formulae

1	a	2	a	3	d	4	d	5	d	6	b	7	a	8	c	9	c	10	b
11	a	12	b	13	c	14	d	15	a	16	b	17	b	18	d	19	b	20	d
21	b	22	b	23	d	24	a	25	a	26	c	27	b	28	b	29	d	30	c
31	c	32	a	33	a	34	c	35	c	36	b	37	d	38	c	39	c	40	a
41	c	42	b	43	b	44	b	45	c	46	b	47	b	48	c	49	a	50	a
51	c	52	d	53	c	54	b	55	d	56	a	57	b	58	b	59	d		

1. a We have $Z^3 + 3iZ^2 - 3Z - i = 0$, i.e. $(Z + i)^3 = 0$. Therefore, $Z = -i$. Hence, the value of the expression $Z^4 + Z^3 + Z^2 + Z + 1 = 1 + i - 1 - i + 1 = 1$.

2. a $e^{a + ib} = e^a(\cos b + i \sin b)$. So the modulus of $e^{a + ib} = e^a$.

3. d Let O be the origin. By problem, P is a point which represents Z. |Z| represents the distance OP. The value of OP will be least, i.e. the value of |Z| will be least if OP becomes the perpendicular from O to the line $3x + 4y = 12$. The length of the perpendicular from O to the line $3x + 4y = 12$ is

$$\left| \frac{(3).(0) + (4).(0) - 12}{\sqrt{9 + 16}} \right| = \frac{12}{5}.$$

The least value of |Z| will be $\dfrac{12}{5}$.

4. d The given equation is $|Z - i| = |Z + i|$.
On taking $Z = x + iy$, it becomes $x^2 + (y - 1)^2 = x^2 + (y + 1)^2$. Therefore, $y^2 - 2y + 1 = y^2 + 2y + 1$, (i.e.) $4y = 0$, (i.e.) $y = 0$.
The condition $|Z - i| = |Z + i| \Rightarrow y = 0$, and x any value. Therefore, the equation will have many solutions.

Alternative method:
Locus of z is the perpendicular bisector of the line segment joining i and –i, i.e. real axis.

5. d
$$\frac{2 + \sqrt{3} + i}{2 + \sqrt{3} - i} = \frac{1 + \dfrac{\sqrt{3}}{2} + \dfrac{i}{2}}{1 + \dfrac{\sqrt{3}}{2} - \dfrac{i}{2}} = \frac{1 + \cos\alpha + i\sin\alpha}{1 + \cos\alpha - i\sin\alpha}$$

$$= \frac{2\cos^2\dfrac{\alpha}{2} + i\, 2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}{2\cos^2\dfrac{\alpha}{2} - 2\, i\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}$$

$$= \frac{e^{i\alpha/2}}{e^{-i\alpha/2}} = e^{i\alpha} = \cos\alpha + i\sin\alpha \quad \text{where } \alpha = \frac{\pi}{6}$$

$$\left(\frac{2 + \sqrt{3} + i}{2 + \sqrt{3} - 1} \right)^{12} = \cos(12\alpha) + i\sin(12\alpha)$$

$$= \cos(2\pi) + i\sin(2\pi) = 1 + i0 = 1$$

Therefore, $a + b = 1 + 0 = 1$

6. b Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = 2\begin{vmatrix} a+b+c & c+a & a+b \\ a+b+c & b+c & c+a \\ a+b+c & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a+b+c & -b & -c \\ a+b+c & -a & -b \\ a+b+c & -c & -a \end{vmatrix}$$

$[C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1] =$

$$2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$[C_1 \to C_1 + C_2 + C_3$ and taking out (-1) common from C_2 and C_3].

7. a Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get $\Delta = \begin{vmatrix} x+1 & x+2 & x+4 \\ 2 & 3 & 4 \\ 6 & 8 & 10 \end{vmatrix} = \begin{vmatrix} x+1 & 1 & 3 \\ 2 & 1 & 2 \\ 6 & 2 & 4 \end{vmatrix}$

$[C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1]$

$$= \begin{vmatrix} x+1 & 1 & 1 \\ 2 & 1 & 0 \\ 6 & 2 & 0 \end{vmatrix} = -2. \quad [C_3 \to C_3 - 2C_2].$$

8. c $A^{-1} = \dfrac{1}{6+1}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \dfrac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

9. c $\text{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Remember: Adjoint of a 2x2 matrix can be obtained by interchanging principal diagonal elements and changing the signs of other two elements.

10. b $P = \dfrac{A + A^T}{2} \left[A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}, \ A^T = \begin{pmatrix} 3 & 1 \\ -5 & -1 \end{pmatrix} \right]$

11. a Suppose x^7 occurs in $(r+1)$th term. Let T_{r+1} denote the $(r+1)$th term in the expansion of $(x - 2x^2)^{-3}$, because $(x - 2x^2)^{-3} = (1 - 2x)^{-3}x^{-3}$. Therefore,

$T_{r+1} = x^{-3} \cdot \dfrac{(-3)\ (-3-1)(-3-2)...(-3-(r-1))}{r!} \ (-2x)^r$

$= x^{-3} \cdot \dfrac{(-3)(-4)(-5)...(-(r+2))}{r!} \ (-1)^r \, 2^r \, x^r$

$= x^{-3} \cdot \dfrac{(-1)^r \, 3 \cdot 4 \cdot 5 \, ...(r+2)}{r!} \ (-1)^r \, 2^r \, x^r$

$= (-1)^{2r} \dfrac{3 \cdot 4 \cdot 5...(r+2)}{r!} \ 2^r \, x^{r-3}$

$= \dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \,...(r+2)}{1 \cdot 2 \cdot r!} \ 2^r \, x^{r-3}$

$= \dfrac{(r+2)!}{2 \times r!} \cdot 2^r x^{r-3} = \dfrac{(r+2)!}{r!} \cdot 2^{r-1} x^{r-3}$ ... (i)

For this term to contain x^7, we must have $r - 3 = 7 \Rightarrow r = 10$.
Putting $r = 10$ in (i), we get

$T_{11} = \dfrac{12!}{10!} \ 2^9 x^7 = 132 \times 2^9 \times x^7$.

Hence, the coefficient of $x^7 = 132 \times 2^9 = 67584$.

12. b We have

$\dfrac{e^{7x} + e^x}{e^{3x}} = e^{4x} + e^{-2x} = \sum\limits_{n=0}^{\infty} \dfrac{(4x)^n}{n!} + \sum\limits_{n=0}^{\infty} \dfrac{(-2x)^n}{n!}$

$= \sum\limits_{n=0}^{\infty} \dfrac{4^n}{n!} \, x^n + \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n \, 2^n}{n!} x^n$

Therefore, the coefficient of x^n in

$\dfrac{e^{7x} + e^x}{e^{3x}} = \dfrac{4^n}{n!} + \dfrac{(-1)^n 2^n}{n!} = \dfrac{2^n}{n!} \left\{ 2^n + (-1)^n \right\}$

13. c $T_n = \dfrac{1 + 3 + 5 + ... n \text{ terms}}{n!} \, x^{n-1}$

$= \dfrac{\frac{1}{2} n\{2.1 + (n-1) \cdot 2\} x^{n-1}}{n!} = \dfrac{n^2 x^{n-1}}{n!}$

$= \dfrac{n x^{n-1}}{(n-1)!} = \dfrac{(n-1)+1}{(n-1)!} \, x^{n-1}$

or $T_n = \dfrac{x^{n-1}}{(n-2)!} + \dfrac{x^{n-1}}{(n-1)!}$.

Putting $n = 1, 2, 3, ...$ and adding column-wise, the sum of the given series

$= \sum\limits_{n=1}^{\infty} \dfrac{x^{n-1}}{(n-2)!} + \sum\limits_{n=1}^{\infty} \dfrac{x^{n-1}}{(n-1)!}$

$= \left[\dfrac{1}{(-1)!} + \dfrac{x}{0!} + \dfrac{x^2}{1!} + \dfrac{x^3}{2!} + ... \right] +$

$\left[\dfrac{1}{0!} + \dfrac{x}{1!} + \dfrac{x^2}{2!} + ... \right] = x \left(1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + ... \right) + e^x$

$= xe^x + e^x = (x + 1) e^x$

14. d Let T_n is the nth term of the given series.

Then $T_n = \dfrac{1 + a + a^2 + ... + a^{n-1}}{n!}, n = 1, 2, 3, ...$

$\Rightarrow T_n = \dfrac{1}{n!} \left[\dfrac{1 - a^n}{1 - a} \right]$

$\Rightarrow T_n = \dfrac{1}{n!} \left[\dfrac{1}{1-a} - \dfrac{a^n}{1-a} \right] = \dfrac{1}{(1-a)} \cdot \dfrac{1}{n!} - \dfrac{1}{(1-a)} \dfrac{a^n}{n!}$

Therefore, $\sum\limits_{n=1}^{\infty} \left[\dfrac{1}{(1-a)n!} - \dfrac{a^n}{(1-a)n!} \right]$

$= \dfrac{1}{(1-a)}(e-1) - \dfrac{1}{(1-a)}(e^a - 1) = \dfrac{e - e^a}{(1-a)} = \dfrac{e^a - e}{a-1}$

15. a Consider the series $\dfrac{3^2}{1!} + \dfrac{5^2}{3!} + \dfrac{7^2}{5!} + ..., \infty$.

Here $t_n = \dfrac{(2n+1)^2}{(2n-1)!}$

$= \dfrac{4n^2 + 4n + 1}{(2n-1)!} = \dfrac{(2n-1)(2n-2) + 10n - 1}{(2n-1)!}$

$= \dfrac{1}{(2n-3)!} + \dfrac{10n-1}{(2n-1)!} = \dfrac{1}{(2n-3)!} + \dfrac{5(2n-1) + 4}{(2n-1)!}$

$$= \frac{1}{(2n-3)!} + 5 \cdot \frac{1}{(2n-2)!} + 4 \cdot \frac{1}{(2n-1)!}$$

If $S = \dfrac{3^2}{1!} + \dfrac{5^2}{3!} + \dfrac{7^2}{5!} + ..., \infty,$

then $S = \displaystyle\sum_{n=1}^{\infty} t_n$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-3)!} + 5 \sum_{n=1}^{\infty} \frac{1}{(2n-2)!} + 4 \sum_{n=1}^{\infty} \frac{1}{(2n-1)!}$$

$$= \left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + ... \right) + 5 \left(1 + \frac{1}{2!} + \frac{1}{4!} + ... \right)$$

$$+ 4 \left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + ... \right)$$

$$= \frac{1}{2} \left(e - e^{-1} \right) + 5 \cdot \frac{1}{2} \left(e + e^{-1} \right) + 4 \cdot \frac{1}{2} \left(e - e^{-1} \right)$$

$$= \frac{5}{2} \left(e - e^{-1} + e + e^{-1} \right) = \frac{5}{2} \cdot 2e = 5e$$

16. b $\displaystyle\lim_{x \to \infty} \left(\frac{x+1}{x+2} \right)^{2x+1} = \lim_{x \to \infty} \left[\left(1 - \frac{1}{x+2} \right)^{-(x+2)} \right]^{-\left(\frac{2x+1}{x+2} \right)}$

$$\lim_{x \to \infty} \left[\left(1 - \frac{1}{x+2} \right)^{-(x+2)} \right]^{-\left(\frac{2+1/x}{1+2/x} \right)} = e^{-2}$$

17. b $\displaystyle\lim_{x \to 0} \frac{(\cos x)^{\frac{1}{2}} - (\cos x)^{\frac{1}{3}}}{\sin^2 x}$

$$= \lim_{x \to 0} \frac{-\frac{1}{2}(\cos x)^{-\frac{1}{2}} \cdot \sin x + \frac{1}{3}(\cos x)^{-\frac{2}{3}} \sin x}{2 \sin x \cos x}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{2}(\cos x)^{-\frac{1}{2}} + \frac{1}{3}(\cos x)^{-\frac{2}{3}}}{2 \cos x}$$

$$= -\frac{1}{2} \left[\frac{1}{2} - \frac{1}{3} \right] = -\frac{1}{12}$$

18. d $\displaystyle\lim_{h \to 0} \frac{f(1-h) - f(1)}{h^3 + 3h}$

$$= \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} \cdot \left(\frac{-1}{h^2 + 3} \right)$$

$$= f'(1) \left(-\frac{1}{3} \right) = \frac{53}{3}$$

19. b

$$\lim_{x \to \infty} \left(\frac{x^2 + 1}{x + 1} - ax - b \right) = \lim_{x \to \infty} \left[\frac{(1-a)x^2 - (a+b)x + 1 - b}{1 + x} \right]$$

For this limits to exist, degree of numerator must be less than or equal to the degree of denominator. Hence a = 1. So now the above limit reduces to

$$\lim_{x \to \infty} \left(\frac{-(1+b)x + 1 - b}{x + 1} \right) = \lim_{x \to \infty} \left(\frac{-(1+b) + \dfrac{(1-b)}{x}}{1 + \dfrac{1}{x}} \right)$$

$$= -(1 + b) = 0 \implies b = -1$$
So a = 1 and b = −1

20. d By definition of rational and irrational numbers.

21. b $f'(5) = \displaystyle\lim_{h \to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h \to 0} \frac{f(5)(f(h) - 1)}{h}$

$$= 2 \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 2.f'(0) = 2.3 = 6$$
As f(5 + 0) = f(5) = f(5).f(0), i.e. 2 = 2f(0)
∴ f(0) = 1.

22. b Let $x^4 = \sin A$
$y^4 = \sin B$
$a(\sin A - \sin B) = \cos A + \cos B$

$$a \, 2 \sin \left(\frac{A-B}{2} \right) \cos \left(\frac{A+B}{2} \right) = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

$$\Rightarrow a \sin \left(\frac{A-B}{2} \right) = \cos \left(\frac{A-B}{2} \right) \Rightarrow \cot \left(\frac{A-B}{2} \right) = a$$

$$\Rightarrow \frac{A - B}{2} = \cot^{-1} a$$

$A - B = 2 \cot^{-1} a$
$\sin^{-1} x^4 - \sin^{-1} y^4 = 2 \cot^{-1} a$

$$\frac{4x^3}{\sqrt{1 - x^8}} - \frac{4y^3}{\sqrt{1 - y^8}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x^3}{y^3} \sqrt{\frac{1 - y^8}{1 - x^8}}$$

23. d $\tan(x + y) + \tan(x - y) = 1$

$$\sec^2(x + y) \left(1 + \frac{dy}{dx} \right) + \sec^2(x - y) \left(1 - \frac{dy}{dx} \right) = 0$$

$$\Rightarrow \sec^2(x + y) + \sec^2(x - y)$$

$$= (\sec^2(x - y) - \sec^2(x + y)) \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2(x + y) + \sec^2(x - y)}{\sec^2(x - y) - \sec^2(x + y)}$$

24. a Putting, $x = \tan\theta$

We get, $\dfrac{\sqrt{1+x^2}-1}{x} = \dfrac{\sec\theta-1}{\tan\theta}$

$= \dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$

So, $y = \tan^{-1}\left(\tan\dfrac{\theta}{2}\right) = \dfrac{\theta}{2} = \dfrac{1}{2}\tan^{-1}x$

So, $y' = \dfrac{1}{2(1+x^2)}$

So, $y'(0) = \dfrac{1}{2}$

25. a $\left(ae^x + 8be^{2x} + 27ce^{3x}\right) - 6\left(ae^x + 4be^{2x} + 9ce^{3x}\right)$

$+ 11\left(ae^x + 2be^{2x} + 3ce^{3x}\right) - 6\left(ae^x + be^{2x} + ce^{3x}\right) = 0$

Alternative solution:
This you will learn in the chapter of 'differential equations', differential equation whose general solution is $y = ae^x + be^{2x} + ce^{3x}$ is given by $(D-1)(D-2)(D-3)y = 0$ where $D = d/dx$.

26. c $\dfrac{7}{3}\displaystyle\int\left(3x - \dfrac{6}{7}\right)\sqrt{3x+2}\ dx$

$= \dfrac{7}{3}\displaystyle\int\left(3x + 2 - 2 - \dfrac{6}{7}\right)\sqrt{3x+2}\ dx$

$= \dfrac{7}{3}\displaystyle\int\left\{3x + 2 - \dfrac{20}{7}\right\}\sqrt{3x+2}\ dx$

$= \dfrac{7}{3}\displaystyle\int(3x+2)^{\frac{3}{2}}\ dx - \dfrac{20}{3}\displaystyle\int\sqrt{3x+2}\ dx$

$= \dfrac{14}{45}(3x+2)^{\frac{5}{2}} - \dfrac{40}{27}(3x+2)^{\frac{3}{2}} + c$

27. b Let $e^x = t$
$e^x dx = dt$

$\displaystyle\int\dfrac{dt}{t^2+6t+5} = \int\dfrac{dt}{(t+3)^2-2^2} = \dfrac{1}{4}\log\left|\dfrac{t+3-2}{t+3+2}\right| + c$

$= \dfrac{1}{4}\log\left|\dfrac{e^x+1}{e^x+5}\right| + c$

28. b Since the integrand is a rational function of $\cos x$,

we put $\tan\left(\dfrac{x}{2}\right) = t$. Then

$I = \displaystyle\int\dfrac{dx}{5+4\cos x} = \int\dfrac{2dt}{(1+t^2)\left(5+\dfrac{4(1-t^2)}{(1+t^2)}\right)}$

$= 2\displaystyle\int\dfrac{dt}{5(1+t^2)+4(1-t^2)}$

$= 2\displaystyle\int\dfrac{dx}{t^2+9} = \dfrac{2}{3}\tan^{-1}\dfrac{t}{3} + C = \dfrac{2}{3}\tan^{-1}\left(\dfrac{1}{3}\tan\dfrac{x}{2}\right) + C$

Hence, $K = \dfrac{2}{3}$ and $M = \dfrac{1}{3}$.

29. d $I = \displaystyle\int_2^4\dfrac{\sqrt{x^2-4}}{x^4}\ dx = \int_2^4\dfrac{1}{x^3}\sqrt{1-\dfrac{4}{x^2}}\ dx$

Putting, $1 - \dfrac{4}{x^2} = t$ and $\dfrac{8}{x^3}dx = dt$, we get

$I = \dfrac{1}{8}\displaystyle\int_0^{3/4}\sqrt{t}\ dt = \dfrac{1}{8}\times\dfrac{2}{3}\left[t^{3/2}\right]_0^{3/4} = \dfrac{1}{12}\times\dfrac{3}{4}\sqrt{\dfrac{3}{4}} = \dfrac{\sqrt{3}}{32}$

30. c Dividing numerator and denominator by $\cos^2\theta$

$\displaystyle\int_0^{\pi/2}\dfrac{\sec^2\theta\ d\theta}{4+9\tan^2\theta}$

Let, $\tan\theta = t$, $dt = \sec^2\theta\ d\theta$

$\displaystyle\int_0^\infty\dfrac{dt}{4+9t^2} = \dfrac{1}{9}\int\dfrac{dt}{\dfrac{4}{9}+t^2} = \dfrac{1}{9}\times\dfrac{1}{\dfrac{2}{3}}\left[\tan^{-1}\dfrac{t}{2/3}\right]_0^\infty$

$= \dfrac{1}{6}\times\dfrac{\pi}{2} = \dfrac{\pi}{12}$

31. c $25 = x^2 + 16 \Rightarrow x = \pm 3.$

32. a Coordinates of point of internal division

$= \left(\dfrac{-1\times 8 + 4\times 2}{10}, \dfrac{2\times 8 + (-5)\times 2}{10}\right)$ or $\left(0, \dfrac{3}{5}\right)$

Coordinates of point of external division

$= \left(\dfrac{2\times 4 - 8(-1)}{2-8}, \dfrac{2(-5) - 8(2)}{2-8}\right)$ or $\left(-\dfrac{8}{3}, \dfrac{13}{3}\right)$

33. a $(x_1 - 3)^2 + (2-4)^2 = 64$ or $(x_1 - 3)^2 = 60$

or $x_1 = 3 \pm 2\sqrt{15}$

34. c Mid-point of (2, 2) and (–4, –4) is (–1, –1).

Length of median $= \sqrt{(5+1)^2 + (-8+1)^2} = \sqrt{85}$

35. c **Short cut:** Use options. Only in option (c), putting this in the given equation of line gives

$\dfrac{1}{a} - \dfrac{2}{b} + \dfrac{1}{c} = 0$ or $\dfrac{1}{a} + \dfrac{1}{c} = \dfrac{2}{b}$ (which implies that a, b and c are in HP).

36. b $\because\ \vec{c}$ and $\vec{d}$ are collinear.

$\therefore\ \vec{c} = k\vec{d}$

$\Rightarrow (p+1)\vec{a} + 2\vec{b} = k\big[(2p-1)\vec{a} - \vec{b}\big]$

$\Rightarrow \big[(p+1) - k(2p-1)\big]\vec{a} + (2+k)\vec{b} = 0$

$\Rightarrow (p+1) - k(2p-1) = 0,\ 2 + k = 0$

[$\because\ \vec{a}, \vec{b}$ are linearly independent]

Hence, $p = \dfrac{1}{5}$

37. d Let x, y be two scalars such that

$\hat{i} + \lambda\hat{j} + 3\hat{k} = x\left(-2\hat{i} + 3\hat{j} - 4\hat{k}\right) + y\left(\hat{i} - 3\hat{j} + 5\hat{k}\right)$

$\Rightarrow \hat{i} + \lambda\hat{j} + 3\hat{k} = (-2x+y)\hat{i} + (3x-3y)\hat{j} + (-4x+5y)\hat{k}$

$\Rightarrow -2x + y = 1;\ 3x - 3y = \lambda$ and $-4x + 5y = 3$

Solving first and third equations, we get

$x = -\dfrac{1}{3}$ and $y = \dfrac{1}{3}$

$\because$ The vectors are coplanar

$\therefore\ 3\times\left(-\dfrac{1}{3}\right) - 3\times\dfrac{1}{3} = \lambda \Rightarrow \lambda = -2$

38. c Let x, y be two scalars such that

$2\hat{i} - \hat{j} + \hat{k} = x\left(\hat{i} + 2\hat{j} - 3\hat{k}\right) + y\left(3\hat{i} + a\hat{j} + 5\hat{k}\right)$

$\Rightarrow 2\hat{i} - \hat{j} + \hat{k} = (x+3y)\hat{i} + (2x+ay)\hat{j} + (-3x+5y)\hat{k}$

$\Rightarrow 2 = x + 3y;\ -1 = 2x + ay;\ 1 = -3x + 5y$

$\Rightarrow$ Solving first and third, we get

$x = \dfrac{1}{2},\ y = \dfrac{1}{2}$

$\therefore$ Given vectors are coplanar.

$\therefore\ -1 = 2\times\dfrac{1}{2} + a\times\dfrac{1}{2} \Rightarrow a = -4$

39. c Let $\vec{p}, \vec{q}$ and $\vec{r}$ be the position vectors of P, Q and R respectively.

Therefore, position vectors of S, T and U are

$\dfrac{\vec{q}+\vec{r}}{2}, \dfrac{\vec{r}+\vec{b}}{2}$ and $\dfrac{\vec{p}+\vec{q}}{2}$ respectively.

$\therefore\ \overrightarrow{PS} + \overrightarrow{QT} + \overrightarrow{RU}$

$= \left(\dfrac{\vec{q}+\vec{r}}{2} - \vec{p}\right) + \left(\dfrac{\vec{r}+\vec{p}}{2} - \vec{q}\right) + \left(\dfrac{\vec{p}+\vec{q}}{2} - \vec{r}\right) = \vec{o}$

40. a Since $\vec{a}, \vec{b}$ and $a\times\vec{b}$ are non-coplanar, therefore

$\vec{r} = x\vec{a} + y\vec{b} + z\left(\vec{a}\times\vec{b}\right)$ for some scalars x, y and z.

Now, $\vec{b} = \vec{r}\times\vec{a} \Rightarrow \vec{b} = \left(x\vec{a} + y\vec{b} + z\left(\vec{a}\times\vec{b}\right)\right)\times\vec{a}$

$= y\left(\vec{b}\times\vec{a}\right) + z\left(\left(\vec{a}\times\vec{b}\right)\times\vec{a}\right)$

$= y\left(\vec{b}\times\vec{a}\right) - z\left(\vec{a}\times\left(\vec{a}\times\vec{b}\right)\right)$

$= y\left(\vec{b}\times\vec{a}\right) - z\left(\left(\vec{a}\cdot\vec{b}\right)\vec{a} - \left(\vec{a}\cdot\vec{a}\right)\vec{b}\right)$

or $\vec{b} = y\left(\vec{b}\times\vec{a}\right) + z\left(\vec{a}\cdot\vec{a}\right)\vec{b}\ \left(\because \vec{a}\cdot\vec{b}=0\right)$

Comparing the coefficients, we get y = 0,

$z = \dfrac{1}{\vec{a}\cdot\vec{a}} = \dfrac{1}{|\vec{a}|^2}$. Putting the values of y and z in (i),

we get $\vec{r} = x\vec{a} + \dfrac{1}{|\vec{a}|^2}\left(\vec{a}\times\vec{b}\right)$.

Short cut: Use options. Cross product of (a) with $\vec{a}$ yields $\vec{b}$.

41. c $\overrightarrow{PQ} = (2\hat{i} + 3\hat{j} - 4\hat{k}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 2\hat{j} - 5\hat{k}$

$\overrightarrow{QR} = (7\hat{i} + 4\hat{j} - 9\hat{k}) - (2\hat{i} + 3\hat{j} - 4\hat{k}) = 5\hat{i} + \hat{j} - 13\hat{k}$

$\overrightarrow{PQ}\times\overrightarrow{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -5 \\ 5 & 1 & 13 \end{vmatrix}$

$= \hat{i}(26+5) - \hat{j}(13+25) + \hat{k}(1-10)$

$= 31\hat{i} - 38\hat{j} - 9\hat{k}$

The unit vector perpendicular to the plane of ΔPQR

$= \dfrac{\overrightarrow{PQ}\times\overrightarrow{QR}}{|\overrightarrow{PQ}\times\overrightarrow{QR}|} = \dfrac{31\hat{i} - 38\hat{j} - 9\hat{k}}{\sqrt{(31)^2 + (-38)^2 + (-9)^2}}$

$= \dfrac{31\hat{i} - 38\hat{j} - 9\hat{k}}{\sqrt{2486}}$

42. b A) $= \sin 2A + \sin 2B + \sin 2C$

$= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C$

$= 2\sin C(\cos(A-B) + \cos C)$

$= 2\sin C\left(2\cos\left(\dfrac{A-B+C}{2}\right)\cos\left(\dfrac{A-B-C}{2}\right)\right)$

$= 2\sin C\left(2\cos\left(\dfrac{\pi}{2}-B\right)\cos\left(\dfrac{\pi}{2}-A\right)\right)$

$= 4\sin A\sin B\sin C$ = (iii)

B) $= \sin A + \sin B + \sin C$

$= 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right) + 2\sin\dfrac{C}{2}\cos\dfrac{C}{2}$

$= 2\cos\left(\dfrac{C}{2}\right)\left(\cos\left(\dfrac{A-B}{2}\right) + \sin\dfrac{C}{2}\right)$

$= 2\cos\left(\dfrac{C}{2}\right)\left(\cos\dfrac{A-B}{2} + \cos\dfrac{A+B}{2}\right)$

$= 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$ = (iv)

C) $= \cos A + \cos B + \cos C$

$= 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2} + 1 - 2\sin^2\dfrac{C}{2}$

$= 1 + 2\sin\dfrac{C}{2}\left(\cos\dfrac{A-B}{2} - \cos\dfrac{A+B}{2}\right)$

$= 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$ = (i)

43. b $\sin 12° \sin 48° \sin 54°$

$= \dfrac{\sin 12° \ \sin(60°-12°) \ \sin(60°+12°)\cos 36°}{\sin 72°}$

$= \dfrac{1}{4}\dfrac{\sin 36° \cos 36°}{\sin 72°} = \dfrac{1}{8}.$

44. b $\dfrac{\sqrt{3}}{2}\cos\theta + \dfrac{1}{2}\sin\theta = \dfrac{\sqrt{2}}{2}$ {dividing by $\sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$}

$\Rightarrow \sin\left(\theta + \dfrac{\pi}{3}\right) = \dfrac{1}{\sqrt{2}} = \sin\left(\dfrac{\pi}{4}\right)$

$\Rightarrow \theta = n\pi + (-1)^n\dfrac{\pi}{4} - \dfrac{\pi}{3}.$

45. c The given relation can be written

$3\cdot\dfrac{2\tan(x/2)}{1+\tan^2(x/2)} + 4\cdot\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = 5$

$\Rightarrow 6\tan\dfrac{x}{2} + 4 - 4\tan^2\dfrac{x}{2} = 5 + 5\tan^2\dfrac{x}{2}$

$\Rightarrow 9\tan^2\dfrac{x}{2} - 6\tan\dfrac{x}{2} + 1 = 0 \Rightarrow \tan\dfrac{x}{2} = \dfrac{1}{3}$

46. b $\tan\theta = \dfrac{-1}{\sqrt{3}} = \tan\left(\pi - \dfrac{\pi}{6}\right),$

$\sin\theta = \dfrac{1}{2} = \sin\left(\pi - \dfrac{\pi}{6}\right)$

and $\cos\theta = \dfrac{-\sqrt{3}}{2} = \cos\left(\pi - \dfrac{\pi}{6}\right)$

Hence, principal value is $\theta = \dfrac{5\pi}{6}.$

47. b $\sin^{-1}\dfrac{3}{5} + \cos^{-1}\dfrac{12}{13} = \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{5}{12}$

$= \tan^{-1}\left[\dfrac{\dfrac{3}{4}+\dfrac{5}{12}}{1-\dfrac{3}{4}\cdot\dfrac{5}{12}}\right] = \tan^{-1}\dfrac{56}{33} = \sin^{-1}\dfrac{56}{65}.$

48. c We have $\cot^{-1}x + \tan^{-1}3 = \dfrac{\pi}{2}$

$\Rightarrow \cot^{-1}x + \tan^{-1}3 = \dfrac{\pi}{2} \Rightarrow \tan^{-1}\dfrac{1}{x} + \tan^{-1}3 = \dfrac{\pi}{2}$
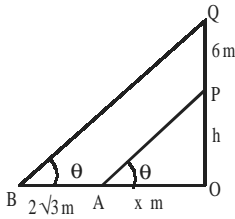
$\Rightarrow \tan^{-1}\left(\dfrac{\dfrac{1}{x}+3}{1-\dfrac{1}{x}\cdot3}\right) = \tan^{-1}\left(\dfrac{1}{0}\right) \Rightarrow \dfrac{3x+1}{x-3} = \dfrac{1}{0} \Rightarrow x = 3.$

Alternative method: As we know that

$\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2},$ therefore for the given

question, x should be 3.

49. a

Let OA and AB be the shadows of tower OP and flagstaff PQ respectively on the grounds.

Suppose the sun makes an angle θ with the ground. Let OA = x.

In triangles OAP and OBQ, we have

$$\tan\theta = \frac{h}{x} \quad \text{and} \quad \tan\theta = \frac{h+6}{x+2\sqrt{3}}$$

$$\therefore \frac{h}{x} = \frac{h+6}{x+2\sqrt{3}} \Rightarrow 2\sqrt{3}h = 6x \Rightarrow x = \frac{h}{\sqrt{3}}.$$

Therefore, $\tan\theta = \frac{h}{x} \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = 60°.$

50. a A committee of 5 persons, consisting of at least two ladies, can be formed in the following ways:

I. Selecting 2 ladies out of 4 and 3 gentlemen out of 6. This can be done in $^4C_2 \times {}^6C_3$ ways.

II. Selecting 3 ladies out of 4 and 2 gentlemen out of 6. This can be done in $^4C_3 \times {}^6C_2$ ways.

III. Selecting 4 ladies out of 4 and 1 gentlemen out of 6. This can be done in $^4C_4 \times {}^6C_1$ ways.

Since, the committee is formed in each case, therefore, by the fundamental principle of addition, the total number of ways of forming the committee

$$= {}^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1$$

$$= 120 + 60 + 6 = 186$$

51. c Since SALIM occupies the second position and the two girls RITA and SITA are always adjacent to each other. So, none of these two girls can occupy the first seat. Thus, first seat can be occupied by any one of the remaining two students in 2 ways. Second seat can be occupied by SALIM in only one way. Now, in the remaining three seats SITA and RITA can be seated in the following four ways:

	I	II	III	IV	V
1.	X	SALIM	SITA	RITA	X
2.	X	SALIM	RITA	SITA	X
3.	X	SALIM	X	SITA	RITA
4.	X	SALIM	X	RITA	SITA

Now, only one seat is left which can be occupied by the 5th student in one way. Hence, the number of required type of arrangements = 2 × 4 × 1 = 8.

52. d There are 5 vowels and 3 consonants in the word 'EQUATION'. Three vowels out of 5 and 2 consonants out of 3 can be chosen in $^5C_3 \times {}^3C_2$ ways. So, there are $^5C_3 \times {}^3C_2$ groups each containing 3 consonants and two vowels. Now, each group contains 5 letters which are to be arranged in such a way that 2 consonants occur together. Considering 2 consonants as one letter, we have 4 letters which can be arranged in 4! ways. But, two consonants can be put together in 2! ways. Therefore, 5 letters in each group can be arranged in $4! \times 2!$ ways. Hence, the required number of words = $(^5C_3 \times {}^3C_2) \times 4! \times 2! = 1440$.

53. c The sample space s can be written
S = {HH, TT, HT, TH}
If E is the event corresponding to occurrence of at least one head, then
n(E) = 3   (HH, HT, TH)
and n(S) = 4

$$\therefore P(E) = \frac{3}{4}$$

54. b We have P (at least one of the events A and B occurs) is 0.6, i.e. $P(A \cup B) = 0.6$ and
P(A and B occur simultaneously) = 0.2, i.e.
$P(A \cap B) = 0.2$.
Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$\Rightarrow 0.6 = P(A) + P(B) - 0.2$
$\Rightarrow 0.6 = 1 - P(\overline{A}) + 1 - P(\overline{B}) - 0.2$
$\Rightarrow 0.6 = 2 - 0.2 - [P(\overline{A}) + P(\overline{B})]$
$\Rightarrow 0.6 = 1.8 - [P(\overline{A}) + P(\overline{B})]$
$\Rightarrow P(\overline{A}) + P(\overline{B}) = 1.8 - 0.6 = 1.2.$

55. d Let E and F be the events defined as follows:
E = Amit solves the problem, F = Bhim solves the problem. Clearly, E and F are independent events

such that $P(E) = \frac{90}{100} = \frac{9}{10}$ and P(F)

$= \frac{70}{100} = \frac{7}{10}$. Now the required probability

$= P(E \cup F) = 1 - P(\overline{E})(\overline{F})$ [As E and F are independent events]

$$= 1 - \left[1 - \left(\frac{9}{10}\right)\right]\left[1 - \left(\frac{7}{10}\right)\right] = 1 - \left(\frac{1}{10}\right) \times \left(\frac{3}{10}\right) = 0.97.$$

56. a Here we are given the mid-values. So we should first find the upper and lower limits of various classes.

The difference between two consecutive values is h = 125 − 115 = 10.

So the lower limit of a class = mid-value − $\dfrac{h}{2}$, and

upper limit = Mid-value + $\dfrac{h}{2}$

Mid-value	Class groups	Frequency	Cumulative frequency
115	110-120	6	6
125	120-130	25	31
135	130-140	48	79
145	140-150	72	151
155	150-160	116	267
165	160-170	60	327
175	170-180	38	365
185	180-190	22	387
195	190-200	3	390
		N = Σf_i = 390	

We have N = 390.

$\therefore \dfrac{N}{2} = \dfrac{390}{2} = 195$.

The cumulative frequency just greater than

$\dfrac{N}{2}$ = 195 is 267 and the corresponding class is

150 – 160.
So 150-160 is the median class.
$\therefore$ l = 150, f = 116, h = 10, F = 151

$\therefore$ Median $= 1 + \dfrac{\dfrac{N}{2} - F}{f} \times h$

$\Rightarrow$ Median $= 150 + \dfrac{195 - 151}{116} \times 10 = 153.8$

57. b Let the frequency of the class 30-40 is f_1 and that of 50-60 is f_2. The total frequency is 229.
$\therefore$ 12 + 30 + f_1 + 65 + f_2 + 25 + 18 = 229
$\Rightarrow f_1 + f_2 = 79$
Median = 46
Clearly, 46 lies in the class 40-50.
So 40-50 is the median class.
$\therefore$ l = 40, h = 10, f = 65 and
F = 12 + 30 + f_1 = 42 + f_1,
N = 229.

Now median $= 1 + \dfrac{\dfrac{n}{2} - F}{f} \times h \Rightarrow 46$

$= 40 + \dfrac{\dfrac{229}{2} - (42 + f_1)}{65} \times 10$

$\Rightarrow f_1 = 34$ (approximately)
Since $f_1 + f_2 = 79$, then $f_2 = 45$.
Hence, $f_1 = 34$ and $f_2 = 45$.
Required sum = $f_1 + f_2 = 79$.

58. b For the given data, $\overline{X}$ = 67.45 in.

Therefore, MD $= \dfrac{\sum f \, |X - \overline{X}|}{N} = \dfrac{226.50}{100} = 2.26$ in

59. d $\overline{X}$ = 67.45 in.

Therefore, $\sigma = \sqrt{\dfrac{\sum f(X - \overline{X})^2}{N}} = \sqrt{8.5275} = 2.92$ in.

60. Let the daily diet consist of x litres of milk, x_2 kg of beef and x_3 dozen of eggs.
$\therefore$ Total cost per day is Rs. z = $15x_1 + 75x_2 + 20x_3$
Total amount of vitamin A in the daily diet is
$x_1 + x_2 + 10x_3$ mg
which should be at least equal to 1 mg.
$\therefore x_1 + x_2 + 10x_3 \geq 1$
Similarly, total amount of vitamins B and C is the daily diet
$100x_1 + 10x_2 + 10x_3 \geq 50$
and $10x_1 + 100x_2 + 10x_3 \geq 10$
Hence, the linear programming formulation to this diet problem is as follows:
Find x_1, x_2, x_3 which minimize z = $15x_1 + 75x_2 + 20x_3$
subject to $x_1 + x_2 + 10x_3 \geq 1$
$100x_1 + 10x_2 + 10x_3 \geq 50$
$10x_1 + 100x_2 = 10x_3 \geq 10$
and $x_1, x_2, x_3 \geq 0$