2011

MATHEMATICS

Paper : 1.1

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

(Objective-type questions)

(*Marks* : 32)

Four keys are provided for the correct answer to each question. Write the correct answer. 2×16=32

1. The geometric series

$$1 + x + x^2 + \cdots \infty \qquad (-1 < x < 1)$$

converges to

(a) $\frac{1}{1+x}$ (b) $\frac{1}{1-x}$

(c) $\frac{1}{x-1}$ (d) $\frac{-1}{1+x}$

2. Which of the following is correct?

   (a) Pointwise convergence $\Rightarrow$ uniform convergence

   (b) Non-uniform convergence $\Rightarrow$ not pointwise convergence

   (c) Uniform convergence $\Leftrightarrow$ pointwise convergence

   (d) Uniform convergence $\Rightarrow$ pointwise convergence

3. The sequence $f_n(x) = nxe^{-nx^2}, x \geq 0$ is uniformly convergent in

   (a) $[0, 1]$

   (b) $[0, 2]$

   (c) $[0, k], k > 0$

   (d) None of the above

4. The series

$$\frac{x}{1+x} + \frac{x}{(1+x)(1+2x)} + \frac{x}{(1+2x)(1+3x)} + \cdots \infty$$

   is

   (a) uniformly convergent on $[a, b], a > 0$

   (b) not uniformly convergent on $[0, b]$

   (c) not pointwise convergent in $[0, b]$

   (d) both uniform and pointwise convergent in $[0, b]$

5. Which of the following is correct statement?

(a) A bounded and monotonic function is not necessarily of bounded variation

(b) A continuous function may not be of bounded variation

(c) A function of bounded variation is not necessarily continuous

(d) A function of bounded variation is not necessarily bounded

6. Total variation of $f(x) = \sin x$ on $[0, \pi/2]$ is

(a) $\pi/2$

(b) 1

(c) $1/2$

(d) -1

7. The value of RS-integral $\int_0^2 x^2 \, dx^2$ is

(a) 2

(b) 3

(c) 4

(d) 8

8. If $f \in R(\alpha)$, then

(a) $\left| \int_a^b f \, d\alpha \right| = \int_a^b |f| \, d\alpha$

(b) $\left| \int_a^b f \, d\alpha \right| > \int_a^b |f| \, d\alpha$

(c) $\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha$

(d) $\left| \int_a^b f \, d\alpha \right| < \int_a^b |f| \, d\alpha$

9. Which of the following is correct?

(a) Measure of a singleton set $\{x\}$ is 1

(b) Measure of a singleton set $\{x\}$ is 0

(c) Measure of $\{2, 4, 6, 8, \cdots\}$ is ∞

(d) Measure of $\{2, 4, 6, 8, \cdots\}$ is 1

10. Which of the following is not measurable?

(a) $f(x) = c \;\; \forall \; x \in \mathbb{R}$

(b) $f + g$ if f and g are measurable

(c) cf if f is measurable

(d) f / g if f and g are any two measurable functions

11. If A is a measurable set, then

(a) $m(A) < m(A\{x\})$

(b) $m(A) = m(A + \{x\})$

(c) $m(A) < m(A) + m\{x\}$

(d) None of the above

12. Which of the following functions is not measurable?

(a) $f(x) = \sin x$ in $[0, \pi/2]$

(b) $f(x) = |x|$ in $[-1, 1]$

(c) $f(x) = |x| - x$ in $[-1, 1]$

(d) $f(x) = \chi_A$, A is a non-measurable set

13. Let

$$f(x) = \begin{cases} 0 & , \text{ when } x \text{ is rational} \\ 1 & , \text{ when } x \text{ is irrational} \end{cases}$$

$f(x)$ is

(a) Riemann integrable but not Lebesgue integrable

(b) Lebesgue integrable but not Riemann integrable

(c) Both Riemann integrable and Lebesgue integrable

(d) Neither Riemann integrable nor Lebesgue integrable

14. The necessary condition for equality of

$$\left| \int_a^b f\, dx \right| = \int_a^b |f\, dx| \text{ is}$$

(a) $f \geq 0$ a.e. or $f \leq 0$ a.e.

(b) $f \neq 0 \ \forall\, x \in [a, b]$

(c) $f > 0 \ \forall\, x \in [a, b]$

(d) $f < 0 \ \forall\, x \in [a, b]$

15. For a bounded function f on $[a, b]$ and partition $P = \{A_1, A_2, \cdots, A_n\}$ of $[a, b]$

(a) $U(P, f) = \sum_{i=1}^{n} \left(\inf_{x \in A_i} f(x) \right) mA_i$ and

$$L(P, f) = \sum_{i=1}^{n} \left(\sup_{x \in A_i} f(x) \right) mA_i$$

(b) $L \int_a^b f\, dx = \inf_P L(P, f)$

(c) $U(P, f) = \sum_{i=1}^{n} \left(\sup_{x \in A_i} f(x) \right) mA_i$ and

$$L(P, f) = \sum_{i=1}^{n} \left(\inf_{x \in A_i} f(x) \right) mA_i$$

(d) $L \int_{-a}^{b} f\, dx = \sup_P U(P, f)$

16. The equality

$$\lim_{n \to \infty} \int_{A \subseteq [a, b]} f_n(x)\, dx = \int_A f(x)\, dx$$

is called

(a) Lebesgue theorem on bounded convergence

(b) Monotone convergence theorem

(c) Lebesgue dominated convergence theorem

(d) Riesz lemma

SECTION—B

(Subjective-type questions)

(*Marks* : 48)

17. Answer any *three* parts : 4×3=12

(a) Show that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{1}{x+n}$$

is uniformly convergent in the interval $[0, b]$, $b > 0$.

(b) Let $\{f_n\}$ be a sequence of functions such that

$$\lim_{n \to \infty} f_n(x) = f(x), \; x \in [a, b]$$

and let $M_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$.

Prove that $f_n \to f$ uniformly on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$.

(c) Test for uniform convergence with the help of Weierstrass's *M*-test of the series

$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \cdots \infty$$

on $[-\frac{1}{2}, \frac{1}{2}]$.

(d) If $\{f_n\}$ be a sequence of continuous functions on an interval $[a, b]$ and if $f_n \to f$ uniformly on $[a, b]$, then prove that f is continuous on $[a, b]$.

(e) Show that the series

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \cdots$$

is not uniformly convergent on $[0, 1]$.

18. Answer any *three* parts : 4×3=12

(a) Show that a monotonic increasing function f in $[a, b]$ is a function of bounded variation. Find the total variation of f.

(b) Let the variation function $v(x)$ of f be continuous at $c \in [a, b]$. Prove that f is continuous.

(c) Determine whether or not the following function f is of bounded variation on $[0, 1]$.

(d) Show that a function f is RS-integrable with respect to α on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U[P, f, \alpha] - L[P, f, \alpha] < \varepsilon$$

(e) Evaluate :

$$\int_1^4 (x - [x])\, dx^2$$

19. Answer any *three* parts : 4×3=12

(a) Define outer measure $m^* A$ and inner measure $m_* A$ of a subset $A \subseteq [a, b]$. Show that

$$m_* A \leq m^* A$$

(b) If A_1 and A_2 are measurable subsets of $[a, b]$, then show that both $A_1 \cup A_2$ and $A_1 \cap A_2$ are measurable.

(c) If f is a measurable function, then show that $|f|$ is also a measurable function.

(d) Show that every continuous function is measurable.

(e) If f is a measurable function, then show that $\{x : f(x) = \alpha\}$ is a measurable set for every $\alpha \in \mathbb{R} U \{\pm\infty\}$.

20. Answer any *three* parts : 4×3=12

(a) If f be a bounded function on $[a, b]$, then show that

$$L \underline{\int_a^b} f\, dx \le L \overline{\int_a^b} f\, dx$$

(b) Give an example of function which is Lebesgue integrable but not Riemann integrable. Justify it.

(c) If f and g are non-negative bounded measurable functions defined on a set $A \subseteq [a, b]$ of finite measure, then show that

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

(d) If f is a Lebesgue integrable function on $[a, b]$, then show that $|f|$ is Lebesgue integrable and

$$\left| \int_a^b f \right| \le \int_a^b |f|$$

(e) State and prove 'Monotone Convergence Theorem'.

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