2011

MATHEMATICS

Paper : 1.3

(Algebra)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

PART—A

(Objective-type Questions)

(*Marks* : 32)

All questions are compulsory. Each question carries four keys *(a)*, *(b)*, *(c)* and *(d)* of which one gives the correct answer. The key giving the correct answer is to be mentioned :

1. G_1 and G_2 are two groups and $G = G_1 \times G_2$ is their direct product. Then G_1 and G_2 are respectively isomorphic to

(a) $A = \{(e_1, a_2) \mid e_1(\text{identity}) \in G_1, a_2 \in G_2\}$
$B = \{(a_1, e_2) \mid e_2(\text{identity}) \in G_2, a_1 \in G_1\}$

(b) $A = \{(e_1, e_2) \mid e_i(\text{identity}) \in G_i\}$
$B = \{(e_1, a_2) \mid e_1(\text{identity}) \in G_1, a_2 \in G_2\}$

(c) $A = \{(e_1, e_2) \mid e_i \text{(identity)} \in G_i\}$
$B = \{(a_1, e_2) \mid e_2 \text{(identity)} \in G_2, a_1 \in G_1\}$

(d) $A = \{(a_1, a_2) \mid a_i \in G_i\}$
$B = \{(e_1, a_2) \mid e_1 \text{(identity)} \in G_1, a_2 \in G_2\}$ 3

2. G is an internal product of its subgroups H_1 and H_2 respectively of G. Then

(a) H_1 is a subgroup of G

(b) H_2 is a subgroup of G, H_1 is a normal subgroup of G

(c) H_1 and H_2 are normal subgroups of G

(d) H_1 is a subgroup of G, H_2 is a normal subgroup of G 2

3. $G = \langle a \rangle$ is a cyclic group of order 6. Then G is an internal direct product of H and K, where

(a) $H = \{e, a, a^2\}, K = \{e, a^3\}$

(b) $H = \{e, a^2, a^4\}, K = \{e, a^3\}$

(c) $H = \{e, a^2, a^4\}, K = \{e, a^2, a^3\}$

(d) $H = \{e, a^4, a^6\}, K = \{e, a^3, a^2\}$ 3

4. The units in the ring $z[i] = \{a + b_i \mid a, b \in z\}$ are

(a) $2 + 2i, 1 - i, +i$

(b) $-1, +2i, -i, +1$

(c) $+1, -1, +i, -i$

(d) $0+3i, -1+i, -1-i, +1-i$ 3

5. In the ring $z[\sqrt{-7}] = \{a+b\sqrt{-7} \mid a, b \in z\}$

(a) $3\sqrt{-7}$ is a prime element

(b) $\sqrt{-5}+\sqrt{-7}$ is a prime element

(c) $\sqrt{-5}$ is a prime element

(d) $\sqrt{-7}$ is a prime element 3

6. In z the ring of integers

(a) the ideal $\langle 27 \rangle$ is a prime ideal

(b) the ideal $\left\langle \frac{3}{7} \right\rangle$ is a prime ideal

(c) the ideal $\langle 13 \rangle$ is a prime ideal

(d) the ideal $\langle 13+5 \rangle$ is a prime ideal 3

7. The ring E of even integers 4 and 6

(a) has LCM but has no HCF

(b) has LCM but has more than one HCF

(c) has no LCM and has no HCF

(d) has no LCM and has HCF 3

8. $Q[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Q\}$ is field extension of Q. The following set is a basis of $Q[\sqrt{2}]$ over Q

(a) $\{\sqrt{2}, -\sqrt{2}\}$ *(b)* $\{1, -\sqrt{2}\}$

(c) $\{1, \sqrt{2}\}$ *(d)* $\{\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \cdots\}$ 3

9. The polynomials $x^2 + 3$ and $x^2 + x + 1$ over Q have same splitting field for

(a) these have no rational roots

(b) one of them have rational roots

(c) both of them have rational roots

(d) none of them have irrational roots 3

10. $\sqrt{2} + \sqrt[3]{7}$ is algebraic over Q of degree

(a) 2 *(b)* 5

(c) 3 *(d)* 6 2

11. α_1, α_2 and α_3 are distinct eigenvalues of a linear transformation $T: V \to V$ (V is a vector space over a field F }. If W_1, W_2 and W_3 are subspaces belonging to α_1, α_2 and α_3 respectively, then

(a) $W_1 \subseteq W_2 + W_3$

(b) $W_1 + W_2 \subseteq W_3$

(c) $W_1 + W_2 + W_3 = 0$

(d) $W_1 \cap (W_2 + W_3) = 0$, $W_2 \cap (W_1 + W_2) = 0$, $W_3 \cap (W_1 + W_3) = 0$ 2

12. $T \in A(V)$ is right invertible if and only if

(a) T has no left inverse

(b) T has negative

(c) T is nilpotent

(d) T is left invertible 2

PART—B

(Subjective-type Questions)

(*Marks* : 48)

Each question carries (6+6=12) marks

Answer any *two* parts of each of the following questions :

13. (a) G_1 and G_2 are two non-trivial groups and $G = G_1 \times G_2$. Prove that—

(i) $G_1^1 = \{(e_1, a_2) \mid a_2 \in G_2\}$ and $G_2^1 = \{(a_1, e_2) \mid a_1 \in G_1\}$ are normal subgroups of G;

(ii) $a_1^1 \, a_2^1 = a_2^1 \, a_1^1$ for $a_1^1 \in G_1^1$, $a_2^1 \in G_2^1$.

(b) x is any element of order mn in a group G such that $(m, n) = 1$. Prove that there exist $y, z \in G$ such that $x = yz = zy$, $o(y) = m$ and $o(z) = n$.

(c) Show that the group S_4 is solvable.

14. *(a)* Prove that in $z[\sqrt{-5}]$, 3 is irreducible but not prime.

(b) R is a PID which is not a field and A is a maximal ideal. Prove that it is generated by an irreducible element.

(c) Show that z is a PID.

15. *(a)* $a \in K$ is algebraic over F. Prove that there is a unique monic polynomial $p(x)$ of positive degree over F such that *(i)* $p(a) = 0$, *(ii)* if for any $f(x) \in F[x]$ $f(a) = 0$, then $p(x)$ divides $f(x)$.

(b) Prove the following :

(i) $x^3 - 2$ is the minimal polynomial of $\sqrt[3]{2}$ over Q

(ii) $x^2 - 3$ is the minimal polynomial of $\sqrt{3}$ over Q

(c) Prove that a finite extension is algebraic. Is the converse true? Justify.

16. *(a)* $T(\neq 0) \in A(V)$, $S \in A(V)$ (with usual notations) are invertible. Prove that T and $S^{-1}TS$ have the same minimal polynomials.

(b) $T(\neq 0) \in A(V)$. Prove the following if υ is an eigenvector of T and $\alpha \in F$ are such that $T(\upsilon) = \alpha\upsilon$, then for any $f(x) \in F[x]$, υ is an eigenvector of $f(T)$ and $[f(T)](\upsilon) = f(\alpha)\upsilon$.

(c) If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are distinct eigenvalues of $T \in A(V)$ and $\upsilon_1, \upsilon_2, \cdots, \upsilon_n$ are non-zero eigenvectors of T belonging to $\alpha_1, \alpha_2, \cdots, \alpha_n$ respectively, then prove that $\upsilon_1, \upsilon_2, \cdots, \upsilon_n$ are LI over F.

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