THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

organised by the SOUTH AFRICAN ACADEMY OF SCIENCE AND ARTS
in collaboration with OLD MUTUAL, AMESA and SAMS

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SECOND ROUND 1998

SENIOR SECTION: GRADES 10, 11 AND 12
(STANDARDS 8, 9 AND 10)

26 MAY 1998

TIME: 120 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. D
2. B
3. B
4. D
5. A
6. D
7. E
8. A
9. A
10. C
11. A
12. C
13. B
14. A
15. D
16. B
17. B
18. D
19. B
20. A

SOLUTIONS

1.

$$\frac{90}{100} \times \frac{80}{100} = \frac{72}{100} = 1 - \frac{28}{100}.$$

Therefore equivalent to a 28% discount.

2. The number of integers is $m - 1 - n = m - n - 1$.

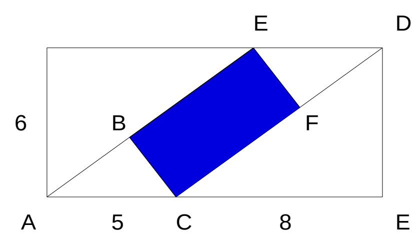
3. The radius of the hosepipe is 7,5mm = 0,75cm, and the length is 20m = 2000cm. It follows that the volume of water in the hosepipe is $\pi(0,75)^2 \times 2000\text{cm}^3$. This is approximately $\frac{22}{7} \times \frac{7}{10} \times \frac{8}{10} \times 2000 = 3520\text{cm}^3$, which is about 3,5 litres.

4. A square with side length 1 has perimeter 4. A square with perimeter $4 \times 4 = 16$ has side length 4. The ratio of their areas is $4^2 : 1$ which is 16:1.

5. $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$. 5 is prime, so one of a or b has to be 5. Then $6! = 6 \times (5!) = (3 \times 2 \times 1) \times (5!)$. So a and b are 3 and 5, and $a + b = 8$.

6. The 15th odd number is 29, and the 15th figure has a total of 29^2 small squares. But each figure contains one more black square than white squares. Therefore the number of black squares is $\frac{1}{2}(29^2 + 1) = 421$.

7. Triangles ABC and CED are similar. By Phythagoras' theorem $CD = 10$. Therefore $AB = 4$ and $BC = 3$. Hence $FC = 10 - 4 = 6$, and the area of $BCFE$ is $3 \times 6 = 18$.



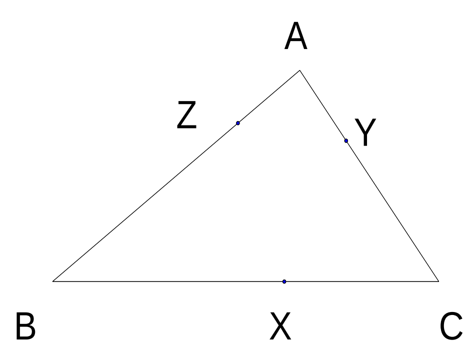
8. Multiplying out $(p-1)(p^{12} + p^{13} + \cdots + p^{41})$ we obtain $p^{42} - p^{12} = (p^6)^7 - (p^6)^2 = 2^7 - 2^2 = 128 - 4 = 124$.

9. Square both sides of the given equation: $m + n + 2\sqrt{mn} = 7 + \sqrt{48} = 7 + 2\sqrt{12}$. Therefore $m + n = 7$ and $mn = 12$. Clearly $m = 3$ and $n = 4$ will work and $m^2 + n^2 = 9 + 16 = 25$.

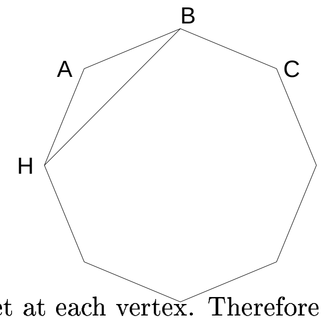
10. Let $CX = CY = x$, $BX = BZ = y$ and $AY = AZ = z$. Then

$$\begin{aligned} y + z &= c \qquad (1) \\ z + x &= b \qquad (2) \\ x + y &= a \qquad (3) \end{aligned}$$

We want to find z. By adding (1) and (2) we get $2z + (x + y) = b + c$. But $x + y = a$ (by (3)). Therefore $2z = b + c - a$.



11. The angle at A is $\frac{1}{8}(8\times 180^o - 360^o) = 180^o - 45^o = 135^o$. But triangle AHB is isosceles. Therefore angle ABH is $\frac{1}{2}(180^o - 135^o) = 22\frac{1}{2}^o$. But angle ABC is also 135^o, so that the angle HBC must be $135^o - 22\frac{1}{2}^o = 112\frac{1}{2}^o$.



12. Each pentagon has 5 vertices, and there are 12 faces. But 3 faces meet at each vertex. Therefore there must be $\frac{1}{3}(5\times 12) = 20$ vertices. Three edges meet at each vertex. But that counts each edge twice. Therefore there are $\frac{1}{2}(3\times 20) = 30$ edges. The total is 20+30 = 50.

13. If C speaks the truth then both A and B are lying. But then B's statement 'Agnes is lying' must be false. Therefore A speaks the truth, which is a contradiction. Therefore C is lying. It follows that either both A and B are speaking the truth or at most one of them is lying.
If A tells the truth then A's statement 'Billy always speaks the truth' must be true. But that contradicts B's statement 'Agnes is lying'. Therefore A is also lying. But A and B do not both lie. Therefore B must speak the truth.
Note: A lies. Therefore B does not always speak the truth; B sometimes lies. Here B happens to be telling the truth.

14. The equation can be written in the form

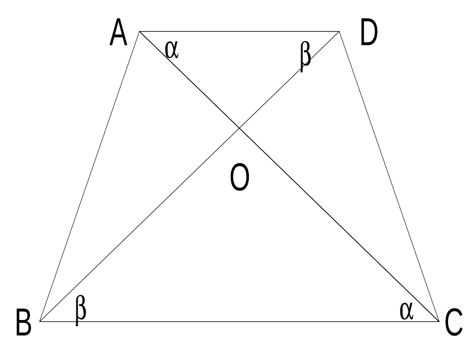
$$3(2x+5y) = 85,$$

where $2x+5y$ is an integer. Clearly 3 is a factor of the left hand side but not of the right hand side. Therefore there can be no solutions.

15. Triangles ADO and CBO are similar. Therefore $\frac{DO}{OB} = \frac{AO}{OC}$. But the areas of triangles with the same height are in the same ratio as the lengths of their bases. So

$$\frac{DO}{OB} = \frac{\text{Area }\triangle AOD}{\text{Area }\triangle AOB} = \frac{AO}{OC} = \frac{\text{Area }\triangle AOB}{\text{Area }\triangle BOC}.$$

Therefore $(\text{Area }\triangle AOB)^2 = xy$, and Area $\triangle AOB$ is $\sqrt{xy}$. Similarly Area $\triangle DOC$ is also $\sqrt{xy}$. (Or by symmetry) Therefore the area of the trapezium is $x + 2\sqrt{x}\sqrt{y} + y = (\sqrt{x}+\sqrt{y})^2$.



16. Let $t = 1 + \frac{1}{9} + \frac{1}{25} + \cdots = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$. Then $s - t = \frac{1}{2^2} + \frac{1}{4^2} + \cdots = \frac{1}{4}(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots) = \frac{1}{4}s$. Therefore $t = \frac{3}{4}s$.

17.

$$\begin{aligned} f(2) &= 2 + f(1) = 2 + 1 \\ f(3) &= 3 + f(2) = 3 + 2 + 1 \\ f(4) &= 4 + f(3) = 4 + 3 + 2 + 1, \text{etc.} \\ f(19) &= 19 + 18 + \cdots + 2 + 1 = \tfrac{1}{2}\cdot 19\cdot(19+1) = 190 \end{aligned}$$

18. If 2 children living in two different houses have to walk to any tree between their houses then the sum of the distances that they walk is equal to the distance between their houses. If these same 2 children have to walk to any tree which is not between their houses then the sum of the distances that they walk is greater than the distance between their houses. Therefore any tree between the 3rd and 4th house gives us the minimum value of the sum of the distances that the 6 children walk. Here there is only one tree, D, between the 3rd and 4th house.

19. Imagine 7 boxes. We have to put one note into each box in order to compose a melody. If all 7 notes were different then we would be able to chose any of 7 notes for the first box, one out of 6 notes for the second box, etc. It follows that we would be able to compose $7 \times 6 \times 5 \times 4 \times 3 \times 3 \times 2$ different melodies. But two of the notes are indistinguishable A's. So they can be interchanged without changing the melody. So we really only have half the number of melodies. The same can be said of the two C's. Finally there are three G's. They can be rearranged in $3 \times 2 \times 1$ different ways without changing the melody. Therefore the total is

1	2	3	4	5	6	7

$$\frac{7 \times 6 \times 5 \times 4 \times 3 \times 3 \times 2}{2 \times 2 \times 3 \times 2} = 120.$$

20. Let us define a new polynomial $Q(x)$ by writing $Q(x) = xP(x) - 1$. This polynomial has degree 1999 and $Q(k) = kP(k) - 1 = 0$ for the 1999 numbers $k = 1, 2, \ldots, 1998, 1999$. Therefore $Q(x) = C(x-1)(x-2)\cdots(x-1998)(x-1999)$, where C is a constant. We can find the value of C as follows. By definition $Q(0) = C(-1)(-2)\cdots(-1998)(-1999) = -C1999!$. Therefore $C = 1/1999!$ and

$$Q(x) = \frac{1}{1999!}(x-1)(x-2)\cdots(x-1998)(x-1999) = xP(x) - 1.$$

Finally put $x = 2000$ and obtain

$$1 = \frac{1999!}{1999!} = 2000P(2000) - 1,$$

and $P(2000) = \frac{1}{1000}$.