12/1

2008

MATHEMATICS

Paper 1

Time : 3 Hours] *[Maximum Marks : 300*

INSTRUCTIONS

Candidates should attempt ***all*** *the questions in Parts A, B & C. However, they have to choose only* ***three*** *questions in Part D.*

Answers must be written in the medium opted (i.e. English or Kannada).

This paper has four parts :

A	20 marks
B	100 marks
C	90 marks
D	90 marks

Marks allotted to each question are indicated in each part.

SEAL

PART A

Each question carries 5 marks. $4 \times 5 = 20$

1. (a) Define concept of the dimension of a vector space and give an example of an infinite dimensional vector space.

(b) Solve $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$, given that $y = 1$ when $x = \sqrt{3}$.

(c) Let f be a function satisfying

$f(x + y) = f(x)\,f(y) - \sqrt{4 - f(y)}$ and $f(h) \to 4$ as $h \to 0$.

Discuss the continuity of f.

(d) If $\vec{A} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{B} = 3\vec{i} - \vec{j} + 2\vec{k}$, then show that $\vec{A} + \vec{B}$ is perpendicular to $\vec{A} - \vec{B}$ and calculate the angle between $2\vec{A} + \vec{B}$ and $\vec{A} + 2\vec{B}$.

PART B

Each question carries 10 marks. *10×10=100*

2. Let W be the subspace of $\mathbf{R}^4$ spanned by the vectors

$$\alpha_1 = (1, 2, 2, 1),\ \alpha_2 = (0, 2, 0, 1) \text{ and } \alpha_3 = (-2, 0, -4, 3).$$

Prove that $\alpha_1, \alpha_2, \alpha_3$ form a basis for W. Also, prove that

$$(1, 0, 2, 0),\ (0, 2, 0, 1),\ (0, 0, 0, 3)$$

form a basis for W.

3. State and prove the Cayley – Hamilton theorem.

4. If $y = (1 + 1/x)^x$, then evaluate $y''(2)$.

5. Determine the points on the curve $5x^2 - 6xy + 5y^2 = 4$ that are nearest the origin.

6. (a) Find the asymptotes of the curve $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$.

(b) Find the minimum value of $x^2 + y^2 + z^2$, given that x, y, z are all positive and $xyz = 8$.

7. A variable line is drawn through O to cut two fixed straight lines L_1 and L_2 in R and S. A point P is chosen on the variable line such that

$$\frac{m+n}{OP} = \frac{m}{OR} + \frac{n}{OS}.$$

Show that the locus of P is a straight line passing through the point of intersection of L_1 and L_2.

8. Solve $x \cos x \dfrac{dy}{dx} + (x \sin x + \cos x)\, y = 1$

9. Solve $(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0$

[Turn over

10. (a) If $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$ and $|\vec{R}| = r$, then show that

$$\text{curl}\left(r^{100}\vec{R}\right) = \vec{0}.$$

(b) Find the work done by the variable force $\vec{F} = 2y\vec{i} + xy\vec{j}$ on a particle, when it is displaced from the origin to the point $\vec{R} = 4\vec{i} + 2\vec{j}$ along the parabola $y^2 = x$.

11. (a) A fluid motion is given by

$$\vec{v} = (y+z)\vec{i} + (z+x)\vec{j} + (x+y)\vec{k}.$$

Is this motion irrotational ? Is the motion possible for an incompressible fluid ?

(b) Evaluate $\iint (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ over the surface of a sphere $x^2 + y^2 + z^2 = 4$.

[redacted]

PART C

Each question carries 15 marks. $6 \times 15 = 90$

12. Let X be a subset of a vector space V over a field F. Prove that X is a basis for V if and only if any mapping of X into any vector space W over F can be uniquely extended to a linear transformation of V into W.

13. (a) Find the coordinates of the point on the curve $x^3 = y(x - a)^2$ whose ordinate is minimum.

(b) If $A > 0$, $B > 0$ and $A + B = \pi/3$, then find the maximum value of $\tan A \tan B$.

14. (a) Evaluate $\iint xy(x + y)\, dx\, dy$ over the area bounded by $y = x^2$ and $y = x$.

(b) By using the transformation $x + y = u$, $y = uv$, evaluate

$$\int_0^1 \int_0^{1-x} e^{y/x+y}\, dx\, dy$$

15. (a) Two circles have centres (a, 0) and (–a, 0) and radii b and c respectively and $a > b > c$. Prove that the points of contact of the common tangents to the two circles lie on $x^2 + y^2 = a^2 \pm bc$.

(b) Circles are drawn through the point (c, 0) touching the circle $x^2 + y^2 = a^2$. If P is a point such that the tangents at the extremities of any chord passing through P intersect on the x-axis. Find the locus of P.

16. Solve the following :

(a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = x \cos x$

(b) $\dfrac{d^2y}{dx^2} + 16y = e^{-3x} + \cos 4x$

17. (a) Show that $g_{ij}\, dx^i\, dx^j$ is an invariant, where g_{ij} is the fundamental covariant function.

(b) If A_{ij} is a skew symmetric tensor, then show that

$$\left(\delta^i_j\, \delta^k_l + \delta^i_l\, \delta^k_j\right) A_{ik} = 0.$$

(c) Prove that the magnitudes of associated vectors are equal.

PART D

Answer any ***three*** *of the following questions. Each question carries 30 marks.* $3 \times 30 = 90$

18. (a) Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, where $4 < a^2 < 8$, that is farthest from the point (0, –2).

(b) Show that the square roots of two successive natural numbers greater than N^2 differ by less than $1/2N$.

(c) State Lagrange's mean value theorem and prove that

$$\frac{x}{1+x} < \log(1 + x) < x \quad \text{for} \quad x > 0.$$

19. (a) Express $\int_0^1 x^m (1 - x^n)^p \, dx$ in terms of gamma function and hence evaluate $\int_0^1 x^5 (1 - x^3)^{10} \, dx$.

(b) Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

(c) Find the centre of gravity of a plane lamina bounded by $r = a(1 + \cos\theta)$.

20. (a) Suppose SY and S′Y′ are the perpendiculars from the foci S and S′ upon the tangent at any point P of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Show that Y and Y′ lie on the circle with the major axis as diameter.

(b) Find the locus of the point of intersection of the tangents to the above ellipse which are at right angles.

(c) Let P be a point on the above ellipse whose ordinate is y'. Prove that the angle between the tangent at P and the focal chord through P is $\tan^{-1}(b^2/aey')$.

21. (a) A uniform rod can move freely about one of its ends and is pulled aside from the vertical by a horizontal force acting at the other end of the rod equal to half its weight. Prove that the rod will rest at an inclination of 45° to the vertical.

(b) In a simple harmonic motion, the distances of a particle from the middle point of its path at three consecutive seconds are observed to be x, y and z. Show that the time of complete oscillation is $2\pi/\cos^{-1}\left(\frac{x+z}{2y}\right)$.

(c) A right circular cone of density ρ floats just immersed with its vertex downwards in a vessel containing two liquids of densities σ_1 and σ_2 respectively. Show that the plane of separation of two liquids cuts off from the axis of the cone a fraction $\left(\frac{\rho-\sigma_2}{\sigma_1-\sigma_2}\right)^{1/3}$ of its length.

SEAL

22. (a) Using divergence theorem evaluate $\int_S \vec{F}\cdot\vec{dS}$ where

$$\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$$

and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

(b) Establish the following :

(i) $[\,ij, k\,] = g_{hk}\begin{Bmatrix} h \\ i\ j \end{Bmatrix}$

(ii) $[\,ij, k\,] + [\,jk, i\,] = \frac{\partial g_{ik}}{\partial x^j}$.

(c) A uniform rod of length l rests in a vertical plane against a smooth horizontal bar at a height h, the lower end of the rod being on the level ground. Show that if the rod is on the point of slipping when its inclination to the horizon is θ, then the coefficient of friction between the rod and the ground is $\frac{l\sin 2\theta\,.\sin\theta}{4h - l\sin 2\theta\cos\theta}$.

2008

MATHEMATICS

Paper 2

Time : 3 Hours] *[Maximum Marks : 300*

INSTRUCTIONS

Candidates should attempt ***all*** *the questions in Parts A, B & C. However, they have to choose only* ***three*** *questions in Part D.*

Answers must be written in the medium opted (i.e. English or Kannada).

This paper has four parts :

A	20 marks
B	100 marks
C	90 marks
D	90 marks

Marks allotted to each question are indicated in each part.

SEAL

PART A

Each question carries 5 marks. *4×5=20*

1. (a) Prove that the elements of a group G which commute with the square of a given element b of G form a subgroup H of G.

   (b) Find $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}}\right)$.

   (c) Find the product of inertia of a rectangular plate of sides a, b about two adjacent sides.

   (d) Apply Runge – Kutta method to find an approximate value of y for $x = 0{\cdot}2$ if $\frac{dy}{dx} = x + y^2$, $y = 1$ when $x = 0$.

PART B

Each question carries 10 marks.   *10×10=100*

2. Show that the set of non-zero residue classes modulo a prime integer p forms an abelian group of order $p-1$ with respect to multiplication of residue classes.

3. If H and K are finite subgroups of a group G, then show that

$$O(HK) = \frac{O(H)\,O(K)}{O(H \cap K)},$$

where O(H) stands for the order of H.

4. (a) If $\{a_n\}$ is monotonically increasing and bounded, then show that $\{a_n\}$ is convergent.

   (b) Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on (0, 1).

5. State and prove Cauchy's integral formula for analytic functions.

6. (a) Form a partial differential equation by eliminating the arbitrary function f from the relation

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

   (b) Solve $q^2y^2 = z(z - px)$. Further, show that there is no singular solution.

7. Find the moment of inertia of a circular plate about a tangent.

8. Describe the procedure to solve an equation by Regula Falsi method. Find the root of the equation $2x - \log_{10} x = 7$ which lies between 3·5 and 4 correct to 5 places of decimals using the method of false position.

9. State and prove Newton's forward interpolation formula.

Using Newton's forward interpolation formula and given the table of values,

x	1·1	1·3	1·5	1·7	1·9
y	0·21	0·69	1·25	1·89	2·61

obtain the values of y, when x = 1·4.

10. Explain Trapezoidal rule for numerical integration.

Evaluate $\int_0^1 \frac{dx}{1+x}$ by Trapezoidal rule by considering eight sub-intervals of the interval [0, 1]. Hence find an approximate of log 2.

11. (a) Employing Euler's method, find the approximate solution of the initial value problem

$$\frac{dy}{dx} = \frac{y-x}{y+x}, \quad y(0) = 1$$

at $x = 0{\cdot}1$, by taking $h = 0{\cdot}02$.

(b) By using the Milne's predictor-corrector method, find an approximate solution of the equation

$$\frac{dy}{dx} = 2\frac{y}{x}, \quad x \neq 0$$

at the point $x = 2$, given that $y(1) = 2$, $y(1{\cdot}25) = 3{\cdot}13$, $y(1{\cdot}5) = 4{\cdot}5$ and $y(1{\cdot}75) = 6{\cdot}13$.

PART C

Each question carries 15 marks. *6×15=90*

12. State and prove Cauchy's theorem for a triangle or a rectangle.

13. Let f be a bounded function defined on E = [a, b] × [c, d]. Let f : E → $\mathbb{R}$ be continuous. Then, show that

$$\iint_E f = \int_a^b \left(\int_c^d f(x, y)dy \right) dx = \int_c^d \left(\int_a^b f(x, y)dx \right) dy$$

14. Show that every integral domain can be imbedded in a field.

15. (a) Find the interpolating polynomial that approximates the function given by

x :	0	1	2	3	4
f(x) :	3	6	11	18	27

(b) A function y = f(x) is specified by the following table :

x :	1	1·2	1·4	1·6	1·8	2·00
y :	0·00	0·128	0·544	1·296	2·432	4·00

Find the approximate values of $f'(1{\cdot}1)$ and $f''(1{\cdot}1)$.

16. If (u, v, w) are orthogonal curvilinear coordinates, then show that $\left(\frac{\partial r}{\partial u}, \frac{\partial r}{\partial v}, \frac{\partial r}{\partial w}\right)$ are reciprocal to the vectors (∇u, ∇v, ∇w), with usual notations.

17. Find the parabola of the form $y = a + bx + cx^2$ which fits most closely with the following observation :

x :	−3	−2	−1	0	1	2	3
y :	4·63	2·11	0·67	0·09	0·63	2·15	4·58

PART D

*Answer any **three** of the following questions. Each question carries 30 marks.* *3×30=90*

18. (a) State and prove Cauchy's residue theorem.

(b) Evaluate $\int_0^{2\pi} \frac{dt}{\sqrt{5} + \cos t}$.

(c) Evaluate $\int_0^{\infty} \frac{dx}{1 + x^4}$.

19. (a) Show that closed subset of a complete metric space is complete.

(b) Show that a metric space is complete if and only if every nested sequence of closed sets $\{F_n\}$ with diameter, $d(F_n) \to 0$ has a non-empty intersection containing precisely one point.

(c) Let (X, d) be a complete metric space. If T is a contraction on X, then show that there is only one x in X, such that T(x) = x.

20. (a) If H is a subgroup of G and N is a normal subgroup of G, then show that $H \cap N$ is a normal subgroup of H.

(b) If a cyclic subgroup N of G is normal in G, then show that every subgroup of N is a normal subgroup of G.

(c) Let G be a group and $a \in G$. Then $N(a) = \{x \in G \mid ax = xa\}$ is a subgroup of G.

(d) Let G be a group and $Z = \{z \in G \mid zx = xz, \forall x \in G\}$. Then show that Z is a normal subgroup of G.

21. (a) Describe Euler's method of solving the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Use the Euler's method to solve the differential equation $y' = -2xy^2$ subject to the condition $y(0) = 1$ for values of x from 0·25 to 1 at steps of length 0·25.

(b) Employing modified Euler's method solve the initial value problem $y' = -y$; $y(0) = 1$ for $x = 0{\cdot}2$ and $0{\cdot}4$.

(c) By Runge – Kutta method of order 4, solve the equation

$$\frac{dy}{dx} = 3x + \frac{y}{2}$$ with $y(0) = 1$ for $y(0{\cdot}2)$, taking step length $h = 0{\cdot}1$.

22. (a) Derive Poisson Distribution from Binomial Distribution.

(b) Find the mean and standard deviation of the Poisson Distribution.

(c) Show that in a Poisson Distribution with Unit Mean, the mean deviation about the mean is $\left[\frac{2}{e}\right]$ times the standard deviation.