Instructions

(1) This question paper consists of 50 multiple choice questions carrying 2 marks each. Answer all questions.
(2) Answers are to be marked in the OMR sheet provided.
(3) For each question, darken the appropriate bubble to indicate your answer.
(4) Use only HB pencils for bubbling answers.
(5) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.
(6) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.
(7) Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ ($\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{Q}_+$ and $\mathbb{C}_+$) denote the set of (respectively positive) integers, real numbers, rational numbers and complex numbers respectively.
(8) For $n \geq 1$, the norm given by $\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ denotes the standard norm on $\mathbb{R}^n$. The metric given by $d(x, y) = \|x - y\|$ is called the standard metric on $\mathbb{R}^n$.

MATHEMATICS

(1) Consider the function

$$f(z) = \frac{1}{1+z^2}$$

where $z \in \mathbb{C}$ and let

$$f(z) = \sum_{n=1}^{\infty} a_n (z-a)^n$$

be the Taylor expansion of $f(z)$ around the point $a \in \mathbb{R}$. The radius of convergence of this power series is

(A) $(1+a^2)^{1/2}$.

(B) $(1+a^2)^{-1/2}$.

(C) $a + (1+a^2)^{1/2}$.

(D) $a - (1+a^2)^{-1/2}$.

(2) Let

$$f : [-1,1] \to \mathbb{R},$$
$$g : [-1,1] \to \mathbb{Q} \cap [-1,1],$$
$$h : \mathbb{R} \to [-1,1]$$

be continuous maps. Then,

(A) both f and g are necessarily not surjective.

(B) both g and h are necessarily not surjective.

(C) both h and f are necessarily not surjective.

(D) all of f, g and h are necessarily not surjective.

(3) Consider the sequence of functions

$$f_n(x) = 1/(1+nx)$$

where $x \in (0,1)$. Then,

(A) $f_n(x) \to 0$ pointwise but not uniformly on $(0,1)$.

(B) $f_n(x) \to 0$ uniformly on $(0,1)$.

(C) $\int_0^1 f_n(x)\,dx \to 0$ as $n \to \infty$.

(D) $f_n'(1/n) \to 0$ as $n \to \infty$.

(4) Let A be a 3×3 matrix with complex entires whose eigenvalues are $1, \pm 2i$. Suppose that for some $\alpha, \beta, \gamma \in \mathbb{C}$,

$$\alpha A^{-1} = A^2 + \beta A + \gamma I$$

where I is the 3×3 identity matrix. Then (α, β, γ) equals

(A) $(-1, -4, 4)$.

(B) $(-4, -1, 4)$.

(C) $(-1, 4, -2)$.

(D) $(-1, -2, 4)$.

(5) Let γ be the circle $|z| = 3$ in the complex plane described in the counterclockwise direction. Then

$$\int_\gamma \frac{3z^2 + z - 2}{(z-2)^2}\, dz$$

equals

(A) $2\pi i$.

(B) $14\pi i$.

(C) $26\pi i$.

(D) $38\pi i$.

(6) Consider the function

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{otherwise.} \end{cases}$$

Then

(A) f is continuous but not differentiable at $x = 0$.

(B) f is differentiable at $x = 0$.

(C) f is continuous but not differentiable at $x = 1$.

(D) f is differentiable at $x = 1$.

(7) Let $p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ be a polynomial of degree of $n \geq 1$ where a_0, a_n are both non-zero. Then

$$f(z) = 1/p(1/z),$$

which is a meromorphic function on $\mathbb{C} \setminus \{0\}$,

(A) has a removable singularity at $z = 0$ and is non-vanishing there.

(B) has a removable singularity at $z = 0$ and has a zero of order n at $z = 0$.

(C) has a pole of order n at $z = 0$.

(D) has an essential singularity at $z = 0$.

(8) Let u, v be eigenvectors of a matrix A corresponding to non-zero real eigenvalues α, β. Suppose that $\alpha \neq \beta$. Then,

(A) $u + v$ is always an eigenvector of A corresponding to $\alpha + \beta$.

(B) $u + v$ is an eigenvector of A only if $\alpha = 0$ and $\beta = 1$.

(C) $u + v$ is an eigenvector of A only if $\alpha = 1$ and $\beta = 0$.

(D) $u + v$ is never an eigenvector of A.

(9) The set of all limit points of

$$S = \{n + \frac{1}{3m^2} : n, m \in \mathbb{N}\}$$

is

(A) $\mathbb{N}$.

(B) $\mathbb{Q}$.

(C) $\mathbb{R}$.

(D) $\mathbb{Z}$.

(10) Let λ be a non-zero real number. Then

$$\lim_{x \to \lambda} \frac{\int_\lambda^x \cos(t^2)\, dt}{x^3 - \lambda^3}$$

equals

(A) $\frac{\cos(\lambda^2)}{3\lambda^2}$.

(B) $\frac{\sin(\lambda^2)}{3\lambda^2}$.

(C) $\frac{2\cos(\lambda^2)}{3\lambda^2}$.

(D) $\frac{2\sin(\lambda^2)}{3\lambda^2}$.

(11) Let $\mathcal{C}^1(\mathbb{R})$ be the collection of all continuously differentiable functions on $\mathbb{R}$. Let

$$S = \{f \in \mathcal{C}^1(\mathbb{R}) : f(0) = 0, f(1) = 1, |f'(x)| \leq 3/4 \text{ for all } x \in \mathbb{R}\}.$$

Then

(A) S is empty.

(B) S is non-empty and finite.

(C) S is countably infinite.

(D) S is uncountable.

(12) Which of the following functions is Lipschitz on $[0, \infty)$?

(A) a polynomial of degree at least 2.

(B) e^x.

(C) $x \sin x$.

(D) the function defined by

$$f(x) = \begin{cases} x^2, & \text{if } 0 \leq x \leq 1; \\ x^{1/2}, & \text{if } 1 \leq x < \infty. \end{cases}$$

(13) Suppose B is a subset of the vector space $\mathbb{R}^3$ with 3 elements.

(A) B must generate $\mathbb{R}^3$.

(B) B cannot be independent.

(C) If B generates $\mathbb{R}^3$ then B is independent.

(D) Either B is independent or B generates $\mathbb{R}^3$.

(14) Let A be an $m \times n$ real-valued matrix and B be an $n \times m$ real-valued matrix so that $AB = I$. Then we must have

(A) $n > m$.

(B) $m \geq n$

(C) if $BA = I$ then $m > n$

(D) either $BA = I$ or $n > m$.

(15) Let A be an $n \times n$ real-valued matrix such that $A^2 = A$.

(A) A must be invertible.

(B) A cannot be invertible.

(C) If A is invertible then $A = I$.

(D) Either $A = I$ or $A = 0$.

(16) Let A be an $n \times n$ real-valued matrix such that $A^2 + I = 0$. Then A cannot be

(A) orthogonal.

(B) skew-symmetric.

(C) symmetric.

(D) invertible.

(17) Let A be a 3×3 real-valued matrix such that $A^3 = I$ but $A \neq I$. Then the trace of A must be

(A) 0.

(B) 1.

(C) -1.

(D) 3.

(18) Which of the following polynomials is reducible over $\mathbb{R}$?

(A) $x^6 + 342x + 18934$.

(B) $x^2 + x + 1$.

(C) $x + 1$.

(D) $x^2 + 2x + 2$.

(19) Let $a, b, c \in \mathbb{Z}$ be integers. Consider the polynomial $p(x) = x^5 + 12ax^3 + 34bx + 43c$.

(A) $p(x)$ is irreducible over $\mathbb{R}$ if and only if $p(x)$ is reducible over $\mathbb{C}$.

(B) $p(x)$ is irreducible over $\mathbb{R}$ if and only if $p(x)$ is irreducible over $\mathbb{Q}$.

(C) $p(x)$ is irreducible over $\mathbb{Z}$ if and only if $p(x)$ is irreducible over $\mathbb{Q}$.

(D) $p(x)$ is irreducible over $\mathbb{Q}$ if and only if $p(x)$ is irreducible over $\mathbb{C}$.

(20) The number of abelian groups of order 27 is

(A) 1.

(B) 2.

(C) 3.

(D) 4.

(21) For which of the following values of n is there a group of order n with no proper normal subgroups?

(A) $n = 21$.

(B) $n = 9$.

(C) $n = 60$.

(D) $n = 98$.

(22) The smallest integer n for which the permutation group S_n on n letters contains an element of order 12 is

(A) 5.

(B) 7.

(C) 9.

(D) 11.

(23) Let G be a finite group of order $3n$ for some $n \in \mathbb{Z}$. Suppose all the elements of G of order 3 are conjugate. Then,

(A) G must be cyclic.

(B) G cannot be abelian.

(C) G must be abelian and not cyclic.

(D) G must be abelian and may or may not be cyclic.

(24) The number of automorphisms (including the identity) of the permutation group S_3 on 3 letters is

(A) 1.

(B) 6.

(C) 9.

(D) 12.

(25) Consider the ring $R = \{a/b : a, b \in \mathbb{Z},\, b \text{ odd}\}$ with the usual addition and multiplication operations. Then,

(A) R is a field isomorphic to $\mathbb{Q}$.

(B) R is a field but not isomorphic to $\mathbb{Q}$.

(C) R is not a field.

(D) R is isomorphic to $\mathbb{Q}$ as a ring but is not a field.

(26) The number of groups of order 6 is

(A) 1.

(B) 2.

(C) 3.

(D) 4.

(27) Let A and B be $n \times n$ real-valued matrices and $C = AB - BA$. Then we must have

(A) $C = 0$.

(B) $C = I$.

(C) $C = -I$.

(D) $C \neq I$.

(28) Suppose X is a subset of $\mathbb{R}$ such that every bounded sequence in X has a subsequence with limit in X. Then,

(A) X must be compact.

(B) X must be open.

(C) X must be closed.

(D) X must be bounded.

(29) For which of the following sets $X \subset \mathbb{R}^2$, with the subspace topology, is there a continuous surjection $f : [0,1] \to X$.

(A) $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y = 0\}$.

(B) $X = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1,\ y = 0\}$.

(C) $X = \{(x, y) \in \mathbb{R}^2 : xy = 0,\ |x| \leq 1,\ 0 \leq y \leq 1\}$.

(D) $X = \{(x, y) \in \mathbb{R}^2 : 1 \leq |x| \leq 2,\ y = 0\}$.

(30) Consider the function

$$f(x) = \begin{cases} \frac{|x|^{\pi-1}}{x}, & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(A) $f(x)$ is continuous everywhere but not differentiable at 0.

(B) $f(x)$ is differentiable everywhere but $f'(x)$ is not continuous at 0.

(C) $f(x)$ is differentiable everywhere and $f'(x)$ is continuous at 0.

(D) $f(x)$ is not continuous at 0.

(31) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x)$ is continuous and $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$.

(A) $f'(x)$ must be bounded.

(B) $f(x)$ must be bounded.

(C) Both $f(x)$ and $f'(x)$ must be unbounded.

(D) Both $f(x)$ and $f'(x)$ must be bounded.

(32) The sequence

$$a_n = (-1)^n \frac{\log(n^4+1)}{n^2+1}$$

(A) is convergent.
(B) is bounded but not convergent.
(C) is neither bounded nor convergent.
(D) is convergent but not bounded.

(33) Let $f : [0,1] \to \mathbb{R}$ be a function such that $g(x) = (f(x))^2$ is continuous. Then,

(A) f must be bounded but need not be continuous.
(B) f must be continuous.
(C) f is continuous if and only if f is bounded.
(D) f is continuous if and only if f is not bounded.

(34) Let (a_n) be the sequence given by

$$a_n = \int_{-\infty}^{\infty} \frac{\cos(nx)}{1+x^2} dx.$$

Then,

(A) (a_n) is bounded.
(B) (a_n) is bounded but does not converge.
(C) (a_n) converges but $\sum_{n=-\infty}^{\infty} a_n$ diverges.
(D) $\sum_{n=-\infty}^{\infty} a_n$ converges.

(35) Let $\varphi, \psi : S \to S$ be two functions on a finite set S such that

$$\varphi(\varphi(x)) = \psi(\psi(x)) = x, \ \forall x \in S.$$

Suppose further that φ has a unique fixed point in S. Then,

(A) ψ must have a unique fixed point.
(B) ψ must have at least one fixed point.
(C) ψ must have no fixed points.
(D) ψ must have an even number of fixed points.

(36) Let u_k and v_k, $k \geq 1$, be real-valued functions satisfying

$$\int_0^1 (u_k(t) + iv_k(t))^4 dt = 0$$

for all k. Let $A_k = \left(\int\limits_0^1 u_k^4(t)dt\right)^{1/4}$ and $B_k = \left(\int\limits_0^1 v_k^4(t)dt\right)^{1/4}$. Then,

(A) A_k/B_k must be bounded but B_k/A_k may be unbounded.

(B) B_k/A_k must be bounded but A_k/B_k may be unbounded.

(C) both A_k/B_k and B_k/A_k must be bounded.

(D) both A_k/B_k and B_k/A_k may be unbounded.

(37) Let (x_n) be a sequence of real numbers which is not Cauchy. Then,

(A) (x_n) is necessarily unbounded.

(B) (x_n) may be convergent.

(C) For every $\epsilon > 0$ there is a subsequence (x_{n_k}) such that $|x_{n_k} - x_{n_j}| < \epsilon$ for all k and j sufficiently large, $k \neq j$.

(D) For some $\epsilon > 0$ there is a subsequence (x_{n_k}) such that $|x_{n_k} - x_{n_j}| > \epsilon$ for all k and j sufficiently large, $k \neq j$.

(38) Let (x_n) be a sequence of complex numbers which converges to 0. Then, we must have,

(A) $\sum\limits_{n=1}^{\infty} x_n$ converges.

(B) $\sum\limits_{n=1}^{\infty} x_n^2$ converges.

(C) There is a subsequence (x_{n_k}) such that $\sum\limits_{n=1}^{\infty} 2^k x_{n_k}$ converges.

(D) There is no subsequence (x_{n_k}) such that $\sum\limits_{n=1}^{\infty} 4^k x_{n_k}$ converges.

(39) Let

$$F(y) = \int\limits_{-\infty}^{\infty} (x+iy)^3 e^{-\frac{(x+iy)^2}{2}} dx, \ y \in \mathbb{R}$$

Then,

(A) $F(y)$ is never 0.

(B) $F(0) = 0$ but $F(y) \neq 0$ for $y \neq 0$.

(C) $F(y) = 0$ for all $y \in \mathbb{R}$.

(D) $F(y) = 0$ if and only if y is rational.

(40) Let $C[0,1]$ be the space of continuous functions on $[0,1]$. Define the operator $T : C[0,1] \to C[0,1]$ by $Tf(x) = f(x^2)$ and let $f_n(x) = T^n f(x)$ for $f \in C[0,1]$. Then,

(A) $g(x) = \lim_{n\to\infty} f_n(x)$ exist for all $x \in [0,1]$ and $g \in C[0,1]$.

(B) $\lim_{n\to\infty} f_n(x)$ need not exist.

(C) $g(x) = \lim_{n\to\infty} f_n(x)$ exist for all $x \in [0,1]$ and $g \in C[0,1]$ if and only if $f(0) = f(1)$.

(D) The sequence $(f_n(x))$ is unbounded for every $x \in [0,1]$.

(41) Let $f \geq 0$ be such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy < \infty.$$

Define $a_{mn} = \int_{m}^{m+1}\int_{n}^{n+1} f(x,y)dxdy$. Then,

(A) $\sum_{n=-\infty}^{\infty} a_{mn}$ converges but $\sum_{m=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} a_{mn}\right)$ diverges.

(B) $\sum_{m=-\infty}^{\infty} a_{mn}$ converges but $\sum_{n=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty} a_{mn}\right)$ diverges.

(C) $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{mn}$ converges.

(D) Both $\sum_{n=-\infty}^{\infty} a_{mn}$ and $\sum_{m=-\infty}^{\infty} a_{mn}$ diverge.

(42) Let $k(\theta) = \sum_{k=-n}^{n} e^{ik\theta}$. Then the value of the integral

$$\frac{1}{2\pi}\int_{0}^{2\pi} \cos(n\varphi)k(\theta-\varphi)d\varphi$$

is

(A) 0.

(B) $\cos(n\theta)$.

(C) $\sin(n\theta)$.

(D) An odd multiple of π.

(43) Suppose f is an entire function. Define

$$\varphi(r) = \sup_{|z|=r} |f(z)|, \ r > 0.$$

Then,

(A) $\int_0^\infty \varphi(r)dr < \infty$ for all entire functions f.

(B) $\int_0^\infty \varphi(r)dr = \infty$ for all entire functions f.

(C) $\int_0^\infty \varphi(r)dr < \infty$ if and only if $f \equiv 0$.

(D) $\int_0^\infty \varphi^2(r)dr < \infty$ for all entire functions f.

(44) Let $v_1, v_2, \ldots, v_m$ be unit vectors in the sphere $S^{n-1} \subset \mathbb{R}^n$ such that $\|v_j - v_k\|^2 = 2$ for $j \neq k$, $1 \leq j, k \leq m$. Then, we must have

(A) m is always greater than n.

(B) m is at most 2^n but may be greater than n.

(C) m is at most n.

(D) m can be infinite.

(45) Let A and B be $n \times n$ real-valued matrices with trace(B) $< 0 <$ trace(A). Then, $F(t) = 1 - det(e^{tA+(1-t)B})$ has

(A) infinitely many zeroes in $0 < t < 1$.

(B) at least one zero in $\mathbb{R}$.

(C) no zeroes.

(D) either no zeroes or infinitely many zeroes in $\mathbb{R}$.

(46) Let A and B be bounded operators on a Hilbert space $\mathcal{H}$ such that $AB = BA$. Let λ be an eigenvalue for A. Then, it must be that

(A) B has no eigenvalues.

(B) B has at least one eigenvalue.

(C) A and B have the same spectrum.

(D) B has empty spectrum.

(47) Let (f_n) be a sequence of entire functions converging to f uniformly on compact subsets of $\mathbb{C}$. Suppose, for all $n \geq 1$, f_n has n zeroes. Then,

(A) f must have infinitely many zeroes.

(B) f need not have any zeroes.

(C) f can have only finitely many zeroes.

(D) f cannot have any zero.

(48) Let $A = \{f \in C[0,1] : f(x) \neq 0 \ \forall x \in [0,1]\}$ where $C[0,1]$ is the set of continuous functions $f : [0,1] \to \mathbb{R}$ with the sup norm. Then,

(A) A is closed.

(B) A is both open and closed.

(C) A is open.

(D) A is neither open nor closed.

(49) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Under which condition does the equation $y'' - y = f$ have a unique solution

(A) $y(0) - y(1) = 1$.

(B) $y(0) = 1 + y(1), \quad y(1) = 2 + y(0)$.

(C) $y'(0) = y'(1) = 0$.

(D) $y(0) = 1 - y'(1)$.

(50) Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex, continuous function such that we have $\lim_{|x| \to \infty} f(x) = \infty$. Then,

(A) f has a unique minimum.

(B) f has a unique maximum.

(C) f has a minimum but it need not be unique.

(D) f has a maximum but it need not be unique.