MATHEMATICS
PAPER & SOLUTION

Code: SS-15-Mathematics

Time : 3 $\frac{1}{4}$ Hours **M.M. 80**

GENERAL INSTRUCTIONS TO THE EXAMINEES:

1. **Candidate must write first his / her Roll No. on the question paper compulsorily.**
2. **All the questions are compulsory.**
3. **Write the answer to each question in the given answer-book only.**
4. **For questions having more than one part, the answers to those parts are to be written together in continuity.**
5. **If there is any error / difference / contradiction in Hindi & English versions of the question paper, the question of Hindi version should be treated valid.**
6.

Section	**Q. Nos.**	**Marks per questions**
A	**1 – 10**	**1**
B	**11 – 25**	**3**
C	**26 – 30**	**5**

7. **There are internal choices in Q. Nos. 11, 12, 15, 17, 29 and 30. You have to attempt only one of the alternatives in these questions.**
8. **Draw the graph of Q. No. 23 on the graph paper**

SECTION - A

1. Find x, if $\tan^{-1}3 + \cot^{-1}x = \frac{\pi}{2}$.

Sol. $\tan^{-1}3 + \cot^{-1}x = \frac{\pi}{2}$

$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}3 = \cot^{-1}3$

$x = 3$

2. Construct a 2 × 2 matrix $A = [a_{ij}]$, whose elements are given by $a_{ij} = |-5i + 2j|$.

Sol. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} |-5+2| & |-5+4| \\ |-10+2| & |-10+4| \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 6 \end{bmatrix}$

3. If $[x \quad -3]\begin{bmatrix} 2x \\ 6 \end{bmatrix} = 0$, then find the value of x.

Sol. $[x \quad -3]\begin{bmatrix} 2x \\ 6 \end{bmatrix} = [2x^2 - 18] = 0$

$2x^2 - 18 = 0$

$x^2 = 9$

$x = \pm 3$

4. Find $\int \frac{\tan x}{\cot x} dx$.

Sol. $\int \frac{\tan x}{\cot x} dx = \int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + c$

5. Find the general solution of the differential equation $\frac{dy}{dx} - \frac{y}{x} = 0$.

Sol. $\frac{dy}{dx} - \frac{y}{x} = 0$

$\int \frac{dy}{y} = \int \frac{dx}{x}$

$\ell ny = \ell nx + c$

$\ell ny - \ell nx = c$

$\ell n \frac{y}{x} = c \Rightarrow \frac{y}{x} = e^c = k$

$y = kx$

6. If $\vec{a} = 2\hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = 4\hat{i} - 2\hat{j} + \lambda\hat{k}$ such that $\vec{a} \parallel \vec{b}$, find the value of λ.

Sol. $\vec{a} \parallel \vec{b}$ $\quad \vec{a} = k\vec{b}$

$(2\hat{i} - \hat{j} + 5\hat{k}) = k(4\hat{i} - 2\hat{j} + \lambda\hat{k})$

$2 = 4k; -1 = -2k; 5 = k\lambda$

$k = \frac{1}{2}$ $\qquad 5 = \frac{1}{2}\lambda$

$\lambda = 10$

7. Find the direction cosine of the line $\frac{x}{4} = \frac{y}{7} = \frac{z}{4}$.

Sol. $\frac{x}{4} = \frac{y}{7} = \frac{z}{4}$ d.r.s $\rightarrow$ 4, 7, 4

d.c.s $\rightarrow \pm \frac{4}{\sqrt{16+49+16}}, \pm \frac{7}{\sqrt{16+49+16}}, \pm \frac{4}{\sqrt{16+49+16}}$

$\pm \frac{4}{9}, \pm \frac{7}{9}, \pm \frac{4}{9}$

8. Find the angle between planes $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$ and $\vec{r}.(2\hat{i} + \hat{j} - \hat{k}) = 7$.

Sol. Angle between planes = angle between their normals

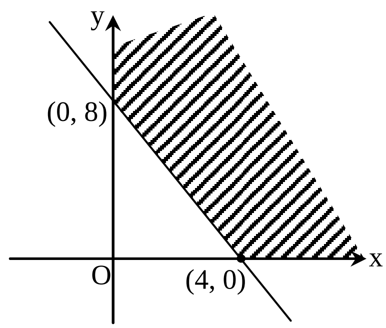
$\cos\theta = \frac{(\hat{i} - \hat{j} + \hat{k})}{\sqrt{3}} . \frac{(2\hat{i} + \hat{j} - \hat{k})}{\sqrt{6}}$

$= \frac{2-1-1}{\sqrt{18}} = 0$

$\theta = 90°$

9. Show the region of feasible solution under the following constraints $2x + y \geq 8$, $x \geq 0$, $y \geq 0$ in answer book.

Sol.



$2x + y \geq 8$; $x \geq 0$; $y \geq 0$

$\frac{x}{4} + \frac{y}{8} \geq 1$

10. If A and B are independent events with P(A) = 0.2 and P(B) = 0.5, then find the value of $P(A \cup B)$.

Sol. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)\,P(B)$

($\because$ A & B are independent)

$= 0.2 + 0.5 - (0.2)(0.5)$

$= 0.7 - 0.10$

$= 0.60$

SECTION - B

11. Prove that the relation R in set of real numbers R defined as $R = \{(a, b) : a \geq b\}$ is reflexive and transitive but not symmetric.

OR

Consider $f : R \to R$ given by $f(x) = 2x + 3$. Show that f is invertible. Find also the inverse of function f.

Sol. $aRb \Rightarrow R = \{(a, b) : a \geq b\}$

$aRa \Rightarrow a \geq a \quad \forall a \in R$ true

so reflexive relation

2R1 $2 \geq 1$ true

but $1 \geq 2 \Rightarrow$ 1R2 not true.

$\therefore$ so not symmetric

$\left.\begin{array}{l} aRb \Rightarrow a \geq b \\ bRc \Rightarrow b \geq c \end{array}\right] a \geq b \geq c$

$\therefore a \geq c \Rightarrow aRc$.

so transitive relation **Hence proved**

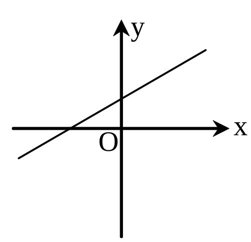
OR

$f : R \to R$ $f(x) = 2x + 3$

$f(x_1) = f(x_2)$

$2x_1 + 3 = 2x_2 + 3$

$x_1 = x_2 \Rightarrow$ so f(x) is one-one function.



$f(x) = 2x + 3$

$x \in R$

$y \in R$

Range = R = codomain

onto function

$\therefore$ f(x) is one-one & onto

$\therefore$ f(x) is invertible function

$y = 2x + 3 \Rightarrow 2x = y - 3$

$x = \frac{y-3}{2}$

Inverse function $f^{-1}(x) = \frac{x-3}{2}$

12. Prove that $\tan^{-1}\left(\frac{2}{9}\right) + \tan^{-1}\left(\frac{1}{4}\right) = \frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$.

OR

Solve $2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$, $0 < x < \frac{\pi}{2}$.

Sol. LHS = $\tan^{-1}\frac{2}{9} + \tan^{-1}\frac{1}{4}$

$$= \tan^{-1}\left(\frac{\frac{2}{9}+\frac{1}{4}}{1-\frac{2}{9}\cdot\frac{1}{4}}\right)$$

$$= \tan^{-1}\left(\frac{8+9}{36-2}\right) = \tan^{-1}\frac{1}{2}$$

$$= \frac{1}{2}\left(2\tan^{-1}\frac{1}{2}\right) = \frac{1}{2}\left(\tan^{-1}\left(\frac{2\cdot\frac{1}{2}}{1-\frac{1}{4}}\right)\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right) = \frac{1}{2}\sin^{-1}\frac{4}{5} = \text{RHS}$$ **Hence proved**

OR

$$2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x) \qquad \left| \; 0 < x < \frac{\pi}{2}\right.$$

$$\tan(2\tan^{-1}(\sin x)) = \tan(\tan^{-1}(2\sec x))$$

$\frac{2\sin x}{1-\sin^2 x} = 2\sec x \quad \Rightarrow \sin x = \sec x.\cos^2 x$

$\Rightarrow \sin x = \cos x$

$\Rightarrow \tan x = 1$

$x = \frac{\pi}{4}$

13. Express the matrix $A = \begin{bmatrix} 2 & -4 & -2 \\ -1 & 4 & 3 \\ 1 & -3 & 2 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Sol. $A = \left(\frac{A+A^T}{2}\right) + \left(\frac{A-A^T}{2}\right)$

$$= \frac{1}{2}\left(\begin{bmatrix} 2 & -4 & -2 \\ -1 & 4 & 3 \\ 1 & -3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ -4 & 4 & -3 \\ -2 & 3 & 2 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 2 & -4 & -2 \\ -1 & 4 & 3 \\ 1 & -3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ -4 & 4 & -3 \\ -2 & 3 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 4 & -5 & -1 \\ -5 & 8 & 0 \\ -1 & 0 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -3 & -3 \\ 3 & 0 & 6 \\ 3 & -6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -5/2 & -1/2 \\ -5/2 & 4 & 0 \\ -1/2 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3/2 & -3/2 \\ 3/2 & 0 & +3 \\ 3/2 & -3 & 0 \end{bmatrix}$$

$\downarrow$ symmetric matrix $\qquad$ $\downarrow$ Skew-symmetrix matrix

14. Find the value of K so that the function is continuous at the point $x = \frac{\pi}{2}$.

$$f(x) = \begin{cases} \dfrac{K\cos x}{\pi - 2x} & ; \quad x \neq \dfrac{\pi}{2} \\ 5 & ; \quad x = \dfrac{\pi}{2} \end{cases}$$

Sol. $$f(x) = \begin{cases} \dfrac{K\cos x}{\pi - 2x} & ; \quad x \neq \dfrac{\pi}{2} \\ 5 & ; \quad x = \dfrac{\pi}{2} \end{cases}$$

$$\text{RHL} = f\left(\frac{\pi^+}{2}\right) = \lim_{x \to \frac{\pi^+}{2}} \frac{K\cos x}{\pi - 2x}$$

put $x = \frac{\pi}{2} + h$

$$= \lim_{h \to 0} \frac{K\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0} \frac{-K\sin h}{-2h}$$

$$= \lim_{h \to 0} \frac{K}{2}\left(\frac{\sin h}{h}\right) = \frac{K}{2}(1) = \frac{K}{2}$$

for function to be continuous at $x = \frac{\pi}{2}$

$$f\left(\frac{\pi^+}{2}\right) = f\left(\frac{\pi}{2}\right)$$

$$\frac{K}{2} = 5 \Rightarrow K = 10$$

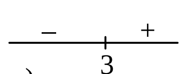
15. Find the intervals in which the function f given by $f(x) = x^2 - 6x + 5$ is
(a) Strictly increasing $\qquad$ (b) Strictly decreasing

OR

Find the equation of the tangent to the curve $x^{2/3} + y^{2/3} = 1$ at the point (1, 1).

Sol. $f(x) = x^2 - 6x + 5$

$f'(x) = 2x - 6 = 2(x - 3)$



(a) strictly increasing $x \in (3, \infty)$

(b) strictly decreasing $x \in (-\infty, 3)$

OR

$x^{2/3} + y^{2/3} = 1$

diff.

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{1}{y^{1/3}}\frac{dy}{dx} = -\frac{1}{x^{1/3}}$$

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

at point (1, 1) $\frac{dy}{dx} = -1$

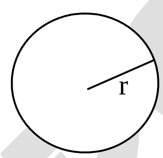
tangent $y - 1 = -1(x - 1)$

$y - 1 + x - 1 = 0$

$x + y - 2 = 0$

16. The radius of a circle is increasing uniformly at the rate of 5cm/sec. Find the rate at which the area of the circle is increasing when the radius is 6 cm.

Sol. $\frac{dr}{dt} = 5$ cm/sec



area = $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$\left(\frac{dA}{dt}\right)_{r = 6cm} = 2\pi(6)(5) = 60\pi \text{ cm/sec}$$

17. Find $\int \frac{(x-1)(x-\log x)^3}{x}dx.$

OR

Find $\int \log(x^2+1)dx.$

Sol. $\int \frac{(x-1)(x-\log x)^3}{x}dx.$

put $x - \ell nx = t$

$$\left(1-\frac{1}{x}\right)dx = dt$$

$$= \int t^3dt = \frac{t^4}{4} + c = \frac{(x-\ell nx)^4}{4} + c$$

OR

$$I = \int \ell n(x^2+1)dx = \int \underset{II}{1}.\underset{I}{\ell n}(x^2+1)dx$$

$$= x\ell n(x^2+1) - \int x \cdot \frac{1}{x^2+1} \cdot 2xdx$$

$$= x\ell n(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}dx$$

$$= x\ell n(x^2+1) - 2\int \left(1-\frac{1}{x^2+1}\right)dx$$

$$= x\ell n(x^2+1) - 2(x - \tan^{-1}x) + c$$

$$= x\ell n(x^2+1) - 2x + 2\tan^{-1}x + c$$

18. Find $\int \frac{1}{3x^2+6x+2}\,dx$.

Sol. $\int \frac{dx}{3x^2+6x+2} = \int \frac{dx}{3x^2+6x+3-1} = \int \frac{dx}{3(x^2+2x+1)-1} = \frac{1}{3}\int \frac{dx}{(x+1)^2-\left(\frac{1}{\sqrt{3}}\right)^2}$

$$= \frac{1}{3}\frac{1}{2\left(\frac{1}{\sqrt{3}}\right)}\ell n\left|\frac{x+1-\frac{1}{\sqrt{3}}}{x+1+\frac{1}{\sqrt{3}}}\right| + c$$

$$= \frac{1}{2\sqrt{3}}\ell n\left|\frac{x+1-\frac{1}{\sqrt{3}}}{x+1+\frac{1}{\sqrt{3}}}\right| + c$$

19. Find the area bounded by the parabola $y^2 = 4x$ and the straight line $y = x$. (Draw the figure in answer book)

Sol. $y^2 = 4x = x^2$

$x = 0 \mid x = 4$

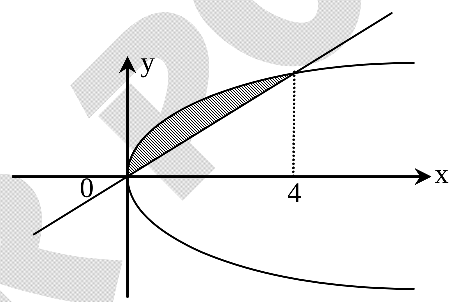
$y_1 = 2\sqrt{x}$; $y_2 = x$

Area $= \int_0^4 (2\sqrt{x} - x)\,dx$

$= 2\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\Big|_0^4$

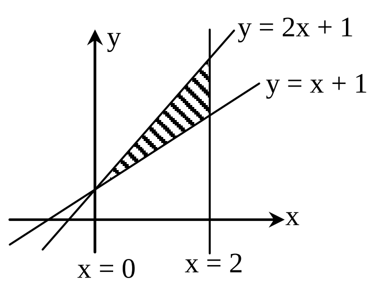
$= \frac{4}{3}(4^{3/2}) - \frac{16}{2} - 0$

$= \frac{8}{3}$ sq. units



20. Using integration find the area of a triangular region whose sides have the equations $y = x + 1$, $y = 2x + 1$ and $x = 2$.
(Draw the figure in answer book.)

Sol.



Shaded Area $= \int_0^2 (y_2 - y_1)\,dx = \int_0^2 (2x+1)-(x+1)\,dx$

$= \int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = \frac{4-0}{2} = 2$ sq. units

21. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

Sol. $\vec{a}+\vec{b}+\vec{c} = 0$

$(\vec{a}+\vec{b}+\vec{c})^2 = 0$

$(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c}) = 0$

$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$1 + 1 + 1 + 2\ (\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a} = -3/2$

22. Find a unit vector perpendicular to each of the vectors $2\vec{a}+\vec{b}$ and $\vec{a}-2\vec{b}$, where $\vec{a}=\hat{i}+2\hat{j}-\hat{k}$, $\vec{b}=\hat{i}+\hat{j}+\hat{k}$.

Sol. $\vec{a}=\hat{i}+2\hat{j}-\hat{k}$ $\quad \vec{b}=\hat{i}+\hat{j}+\hat{k}$

$2\vec{a}+\vec{b} = 2\hat{i}+4\hat{j}-2\hat{k}+\hat{i}+\hat{j}+\hat{k}$

$= 3\hat{i}+5\hat{j}-\hat{k}$

$\vec{a}-2\vec{b} = \hat{i}+2\hat{j}-\hat{k}-2\ (\hat{i}+\hat{j}+\hat{k})$

$= -\hat{i}-3\hat{k}$

Required vector = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & -1 \\ -1 & 0 & -3 \end{vmatrix}$

$= \hat{i}\ (-15) - \hat{j}\ (-9-1) + \hat{k}\ (0+5)$

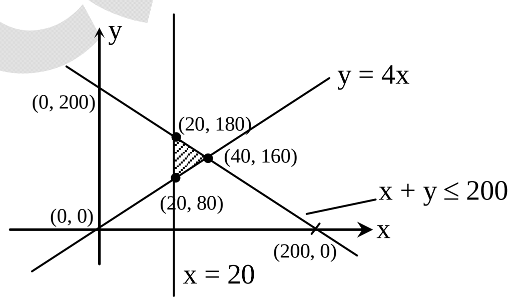
$= -15\hat{i}+10\hat{j}+5\hat{k}$

$= 5\ (-3\hat{i}+2\hat{j}+\hat{k})$

Perpendicular unit vector = $\pm\dfrac{(-3\hat{i}+2\hat{j}+\hat{k})}{\sqrt{9+4+1}} = \dfrac{\pm(-3\hat{i}+2\hat{j}+\hat{k})}{\sqrt{14}}$

23. By graphical method solve the following linear programming problem for maximization.
Objective function Z = 1000x + 600y
Constraints $x + y \le 200$
$4x - y \le 0$
$x \ge 20, \quad x \ge 0, y \ge 0.$

Sol.



Shaded region is required region

Now, Z = 1000x + 600y

x	y	z
20	80	68000
20	180	1,28,000
40	160	1,36,000

$\because Z_{max} = 1,36,000$

When x = 40, y = 160

24. Bag A contains 2 red and 3 black balls while another bag B contains 3 red and 4 black balls. One ball is drawn at random from one of the bag and it is found to be red. Find the probability that it was drawn from bag B.

Sol. Bag A — 2 Red 3 Black; Bag B — 3 Red 4 Black

Event $E_1 \rightarrow$ Bag A is selected $P(E_1) = 1/2$

Event $E_2 \rightarrow$ Bag B is selected $P(E_2) = 1/2$

Event X $\rightarrow$ drawn ball is Red

$P(X) = P(E_1)\, P(X/E_1) + P(E_2)\, P(X/E_2)$

$$P(X) = \frac{1}{2}\left(\frac{2}{5}\right) + \frac{1}{2}\left(\frac{3}{7}\right)$$

$$P(E_2/X) = \frac{P(E_2)P(X/E_2)}{P(X)} = \frac{\frac{1}{2}.\frac{3}{7}}{\frac{1}{2}.\frac{2}{5}+\frac{1}{2}.\frac{3}{7}}$$

$$= \frac{3}{14\left(\frac{1}{5}+\frac{3}{14}\right)} = \frac{3}{14\left(\frac{14+15}{14\cdot 5}\right)} = \frac{15}{29}$$

25. From a lot of 30 bulbs which include 6 defectives, a sample of 2 bulbs are drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

Sol. 30 bulbs 6 defective

let no. of defective bulbs = X

X_i	P_i
0	$\frac{^{24}C_2}{^{30}C_2} = \frac{92}{145}$
1	$\frac{^{6}C_1.\,^{24}C_1}{^{30}C_2} = \frac{48}{145}$
2	$\frac{^{6}C_2}{^{30}C_2} = \frac{1}{29}$

SECTION - C

26. Prove that $\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix}$ = (a + b + c) (a – b) (b – c) (c – a).

Sol. $\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix}$

$C_1 \to C_1 + C_3$

$= \begin{vmatrix} a+b+c & a^2 & b+c \\ b+c+a & b^2 & c+a \\ c+a+b & c^2 & a+b \end{vmatrix}$

$= (a+b+c)\begin{vmatrix} 1 & a^2 & b+c \\ 1 & b^2 & c+a \\ 1 & c^2 & a+b \end{vmatrix}$

Apply $R_1 \to R_1 - R_3$

$R_2 \to R_2 - R_3$

$= (a+b+c)\begin{vmatrix} 0 & a^2-c^2 & c-a \\ 0 & b^2-c^2 & c-b \\ 1 & c^2 & a+b \end{vmatrix}$

$= (a + b + c) [(a^2 - c^2)(c - b) - (c - a)(b^2 - c^2)]$

$= (a + b + c) [(a - c) (a + c) (c - b) - (c - a)(b - c)(b + c)]$

$= (a + b + c) (b - c) (c - a) [(a + c) - (b + c)]$

$= (a + b + c) (a - b) (b - c) (c - a)$ **Hence Proved**

27. If $y = (\sin^{-1}x)^2$, then show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2 = 0$.

Sol. $y = (\sin^{-1} x)^2$

$\dfrac{dy}{dx} = 2\sin^{-1}x \times \dfrac{1}{\sqrt{1-x^2}}$ … (1)

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{1-x^2} \times \dfrac{1}{\sqrt{1-x^2}} - 2\sin^{-1}x \cdot \dfrac{1}{2}\dfrac{(-2x)}{\sqrt{1-x^2}}}{(1-x^2)}$$

$(1 - x^2)\dfrac{d^2y}{dx^2} = 2 + \dfrac{x(2\sin^{-1}x)}{\sqrt{1-x^2}}$

$(1 - x^2)\dfrac{d^2y}{dx^2} = 2 + \dfrac{x\,dy}{dx}$ (using 1)

$(1 - x^2)\dfrac{d^2y}{dx^2} - \dfrac{x\,dy}{dx} - 2 = 0$ **Hence proved**

28. Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$.

Sol. $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ … (1)

$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$ $\left\{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right\}$

$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$ … (2)

Add (1) & (2)

$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

$2I = 2\pi \int_0^{\pi/2} \frac{\sin x\, dx}{1+\cos^2 x}$ $\left\{\begin{array}{l}\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \\ \text{if } f(2a-x) = f(x)\end{array}\right\}$

$I = \pi \int_0^{\pi/2} \frac{\sin x\, dx}{1+\cos^2 x}$

Let cos x = t

$\Rightarrow -\sin x\, dx = dt$

$I = \pi \int_1^0 \frac{-dt}{1+t^2}$

$I = -\pi\, [\tan^{-1} t]_1^0$

$I = -\pi\, [\tan^{-1} 0 - \tan^{-1} 1]$

$= -\pi\left(-\frac{\pi}{4}\right) = \frac{\pi^2}{4}$

29. Solve the differential equation $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$.

OR

solve the differential equation $\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x$.

Sol. $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$

$2x^2.\frac{dy}{dx} = 2xy + y^2$

$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{2}\left(\frac{y}{x}\right)^2$ … (1)

Let y = vx

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Put in (1)

$v + \frac{x\,dv}{dx} = v + \frac{1}{2}v^2$

$\frac{x\,dv}{dx} = \frac{v^2}{2}$

$\frac{dv}{v^2} = \frac{dx}{2x}$

Integrating both sides

$\int \frac{dv}{v^2} = \int \frac{dx}{2x}$

$-\frac{1}{v} = \frac{1}{2}\log_e x + c$

$-\frac{x}{y} = \frac{1}{2}\log_e x + c$ Where c is constant of integration

OR

$\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x$ … (1)

I.F. $= e^{\int \cot x\, dx} = e^{\log_e \sin x} = \sin x$

Multiply both sides of by I.F. & then integrate we get

y.sinx $= \int 2x \sin x\, dx + \int x^2 \cot x.\sin x\, dx$

$y \sin x = 2\int x \sin x\, dx + \int x^2 \cos x\, dx$

$= 2\int x \sin x\, dx + x^2 \sin x - \int 2x \sin x\, dx + c$

$y \sin x = x^2 \sin x + c$

30. Find the shortest distance between the lines $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-1}{1}$ and $\frac{x-2}{2} = \frac{y+1}{1} = \frac{z+1}{2}$.

OR

Prove that if a plane has the intercepts a, b, c and is at a distance p units from the origin, then prove that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

Sol. $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-1}{1} \Rightarrow (\vec{r} - \underbrace{(\hat{i} + 2\hat{j} + \hat{k})}_{\vec{a}}) . \underbrace{(\hat{i} - \hat{j} + \hat{k})}_{\vec{p}} = 0$

$\frac{x-2}{2} = \frac{y+1}{1} = \frac{z+1}{2} \Rightarrow (\vec{r} - \underbrace{(2\hat{i} - \hat{j} - \hat{k})}_{\vec{b}}) . \underbrace{(2\hat{i} + \hat{j} + 2\hat{k})}_{\vec{q}} = 0$

Shortest distance $= \left| \frac{(\vec{b} - \vec{a}).(\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|} \right|$

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = \hat{i}(-2-1) - \hat{j}(2-2) + \hat{k}(1+2) = -3\hat{i} + 3\hat{k}$$

$\vec{b} - \vec{a} = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k})$

$= \hat{i} - 3\hat{j} - 2\hat{k}$

Shortest distance $= \left| \frac{(\hat{i} - 3\hat{j} - 2\hat{k}).(-3\hat{i} + 3\hat{k})}{3\sqrt{2}} \right|$

$= \frac{|-3+0-6|}{3\sqrt{2}} = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}}$ units

OR

Plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Perpendicular distance from origin (0, 0, 0)

$$P=\frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$P=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$$ **Hence Proved**