

Total No. of Questions : 8]

SEAT No. :

P1226

[Total No. of Pages : 2

[5121]-11

M.A./M.Sc. (Semester - I)

MATHEMATICS

MT - 501 : Real Analysis

(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicate full marks.

Q1) a) Let $B[a,b]$ be the set of all real valued bounded functions defined on $[a,b]$. For f and $g \in B[a,b]$ show that, d is a metric

$$\text{Where } d(f,g) = \sup_{t \in (a,b)} |f(t) - g(t)| \quad [6]$$

b) If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space then prove that

$$|\langle v, w \rangle| \leq \sqrt{\langle v, v \rangle} \cdot \sqrt{\langle w, w \rangle} \text{ for } v, w \in V \quad [5]$$

c) Show that \mathbb{R}^n is a finite dimensional space with basis $\{e_1, e_2, \dots, e_n\}$. Also state dimensions of $l^1, l^\infty, e[a,b]$. [5]

Q2) a) Let $C[0,1]$ be the family of all continuous, real valued functions defined on $[0,1]$ with supnorm.

$$\text{Find } d(f,g) \text{ where } \langle f, g \rangle = \int_0^1 f(x)g(x)dx \text{ with } f(x) = 1 \text{ and } g(x) = x \quad \forall x \in [0,1] \quad [6]$$

b) Let (M,d) be a metric space and E be a compact subset of M . Prove that E is closed. [5]

c) Define a bounded set and to tally bounded set. Show that a bounded set may not be totally bounded. [5]

P.T.O.

- Q3)** a) State and prove Arzela Ascoli theorem. [6]
 b) Show that real numbers with discrete metric is not separable. [5]
 c) State whether $P[a,b]$ and $C[a,b]$ are separable? Justify. [5]
- Q4)** a) Let ε be the collection of all finite unions of disjoint intervals in \mathbb{R} . If $A \in \varepsilon$ and $\epsilon > 0$. Then show that \exists a closed set $F \in \varepsilon$ and an open set $G \in \varepsilon$ such that $F \subseteq A \subseteq G$, $m(F) \geq m(A) - \epsilon$ and $m(G) \leq m(A) + \epsilon$. [6]
 b) With usual notations prove that, m^* is additive on \mathcal{F} . [5]
 c) Show that if f and g are measurable functions then $f \cdot g$ is also measurable. [5]
- Q5)** a) State and prove Lebesgue monotone convergence theorem. [6]
 b) Give an example of Lebesgue integrable function which is not Riemann integrable. [5]
 c) Show by an example that strict inequality hold's in Fatou's lemma. [5]
- Q6)** a) Show that $L^p(\mu)$ is complete. [6]
 b) State and prove Holder's inequality [5]
 c) Whether step functions are dense in $L^1(\mu)$? Justify. [5]
- Q7)** a) State and prove Bessel's inequality. [8]
 b) Show that the trigonometric system

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin mx}{\sqrt{\pi}}, n = m = 1, 2, \dots$$
 is an orthonormal sequence in $L^2([-\pi, \pi], m)$. [8]
- Q8)** a) State and prove Parseval's theorem [8]
 b) Show that $L^p(\mu)$, $1 \leq p \leq \infty$ is a complete space. [8]



Total No. of Questions : 8]

SEAT No. :

P1227

[Total No. of Pages : 3

[5121]-12
M.A./M.Sc. (Semester - I)
MATHEMATICS
MT - 502 : Advance Calculus
(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) *Attempt any five questions.*
- 2) *Figures to the right indicate full marks.*

- Q1)** a) Define directional derivative of a scalar field. Show that the existence of all directional derivative at a point need not imply the continuity of a function at that point. **[6]**
- b) Give an example of a function $f(x,y)$ of two variables, which is continuous in each variable separately, but is discontinuous as a function of two variables together. **[5]**
- c) If $\bar{f} : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $\bar{f}(\bar{x}) = \|\bar{x}\|^4$, Find all points (x,y,z) for which $f'((1,2,3), (x,y,z)) = 0$. **[5]**
- Q2)** a) State only the chain rule for derivatives of vector fields in matrix form and explained the terms involved. **[6]**
- b) The substitution $u = \frac{x-y}{2}$ $v = \frac{x+y}{2}$ changes $f(u,v)$ into $f(x,y)$. Express the partial derivative $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of the partial derivatives $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$. **[5]**
- c) Find the directional derivative of the scalar field $f(x,y) = x^2 - 3xy$ along the parabola $y = x^2 - x + 2$ at the point $(1,2)$. **[5]**

P.T.O.

- Q3)** a) Define line integral. State only the basic properties of line integral. [6]
- b) Compute the mass M of one coil of a spring having the shape of the helix whose vector equation is $\vec{\alpha}(t) = a \cos t \vec{i} + a \sin t \vec{j} + bt\vec{k}$ if the density at (x,y,z) is $x^2 + y^2 + z^2$. [5]
- c) Calculate the line integral of the vector field $\vec{f}(x, y, z) = x\vec{i} + y\vec{j} + (xz - y)\vec{k}$ from $(0,0,0)$ to $(1,2,4)$ along a line segment. [5]
- Q4)** a) State and prove first fundamental theorem for line integrals. [8]
- b) Find the amount of work done by the force $\vec{f}(x, y) = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in moving a particle (in a clockwise direction) once around the square bounded by the co-ordinate axes and the line $x = a$ and $y = a$, $a > 0$. [5]
- c) Determine whether or not the vector field $\vec{f}(x, y) = 3x^2y\vec{i} + x^3y\vec{j}$ is a gradient on any open subset of \mathbb{R}^2 ? [3]
- Q5)** a) State and prove Green's theorem for plane regions bounded by piecewise smooth Jordan curves [8]
- b) Transform the integral to polar co-ordinates and compute its value
- $$\int_0^{2a} \left[\int_0^{\sqrt{2ax-x^2}} (x^2 + y^2) dy \right] dx. \quad [5]$$
- c) Evaluate the line integral $\int_C (5 - xy - y^2) dx - (2xy - x^2) dy$, where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, traversed counter clockwise. [3]
- Q6)** a) Let $\vec{f}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ be a vector field that is continuously differentiable on an open simply connected set S in the plane. Prove that \vec{f} is gradient on S if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on S . [6]

b) Evaluate integral using polar co-ordinate $\iint_S \sqrt{a^2 - x^2 - y^2} \, dx \, dy$ where the region S is the first quadrant of the circular disk $x^2 + y^2 \leq a^2$. [5]

c) Evaluate $\iiint_S xy^2z^3 \, dx \, dy \, dz$ where S is the solid by the surface $z = xy$ and the planes $y = x$, $x = 1$, and $z = 0$. [5]

Q7) a) Let $\bar{r}(T)$ be smooth parametric surface C^* be a smooth curve in T and $C = \bar{r}(C^*)$ is a smooth curve lying on the surface. Prove that at each point of C the fundamental vector product is normal to C. [6]

b) Compute the surface area of a hemisphere of radius 'a' using surface integral. [6]

c) Find the fundamental vector product for the surface with explicit representation. [4]

Q8) a) State the stokes' theorem and show that the surface integral which appears in stokes' theorem in terms of the curl of a vector field. [6]

b) Let S be the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$ and let \bar{n} be the unit outer normal to S. If $\bar{F}(x, y, z) = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ use the divergence theorem to evaluate the surface integral. [5]

c) Determine the jacobian matrix and compute divergence and curl of \bar{F} , where $\bar{F}(x, y, z) = xy^2z^2\bar{i} + z^2 \sin y\bar{j} + x^2e^y\bar{k}$ [5]



Total No. of Questions : 8]

SEAT No. :

P1228

[Total No. of Pages : 3

[5121]-13
M.A./M.Sc. (Semester - I)
MATHEMATICS
MT - 503 : Linear Algebra
(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) *Solve any five questions.*
- 2) *Figures to the right indicate marks.*

- Q1)** a) Prove that a linearly independent subset of a finite dimensional vector space can be extended to a basis of the vector space. [6]
- b) Let V and V' be vector spaces over a field K of dimensions m and n , respectively. Show that the dimension of the vector space $L(V, V')$ is mn . [5]
- c) Let $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ be defined by $T(p(x)) = xp(x)$. Show that T is a linear operator on $\mathbb{R}[x]$. Also show that if D is the differential operator on $\mathbb{R}[x]$, then $DT - TD = I$. [5]
- Q2)** a) Let V be a finite dimensional vector space over a field K and let W be a subspace of V . Then prove that $\dim V = \dim W + \dim V/W$. [6]
- b) Let T be a linear operator on a finite dimensional vector space such that $T^2 = T$. Prove that $V = \ker T \oplus \text{im} T$. [5]
- c) Let T be a linear operator on a finite dimensional vector space such that $\text{rank } T = \text{rank } T^2$. Show that $\text{im } T \cap \ker T = \{0\}$. [5]
- Q3)** a) Let V_1, V_2, \dots, V_m be vector spaces over a field K . Prove that $V = V_1 \oplus \dots \oplus V_m$ is a finite dimensional if and only if each V_i is finite dimensional. [6]

P.T.O.

- b) Let D be the differential operator on the vector space $\mathbb{R}_4[x]$. Find the matrix of D with respect to the basis $\{1, x, x^2, x^3, x^4\}$. [5]
- c) Let T be a nilpotent operator on a finite dimensional vector space. Show that $\det T = 0$. Is converse true? Justify. [5]
- Q4)** a) State and prove primary decomposition theorem. [6]
- b) Let A and B be similar matrices over \mathbb{R} . Show that B and A' have same characteristic polynomials. [5]
- c) Let A be an $n \times n$ matrix over \mathbb{Q} and let p be a prime, $p > n + 1$. Show that if $A^p = I_n$, then $A = I_n$. [5]
- Q5)** a) Prove that the geometric multiplicity of an eigenvalue of a linear operator can not exceed its algebraic multiplicity. [6]
- b) Let T be a linear operator on a finite dimensional vector space V over K . Prove that if the characteristic polynomial of T splits over K , then T is triangulable. [5]
- c) Give an example of a 3×3 matrix which is not triangulable over \mathbb{R} but is diagonalizable over \mathbb{C} . [5]
- Q6)** a) Let V be a finite dimensional vector space over K and let T be a linear operator on V . Prove that V is a direct sum of T -cyclic subspaces. [6]
- b) Write all possible Jordan canonical forms of a matrix whose characteristic polynomial is $(x - 2)^4 (x - 3)^2$. [5]
- c) Give all possible rational canonical forms of a matrix A whose characteristic polynomial is $(x^2 + 2)(x - 3)^2$. [5]
- Q7)** a) Let V be an inner product space and $u, v \in V$. Show that [6]
- i) $\|u + v\| \leq \|u\| + \|v\|$ and
- ii) $|\|u\| - \|v\|| \leq \|u - v\|$.

- b) Let $x \in \mathbb{R}^n$. Find all eigenvalues of the $n \times n$ matrix xx^* . [5]
- c) Let V be an inner product space and let $\{x_1, x_2, \dots, x_n\}$ be linearly independent in V . Prove that there exists a sequence of orthonormal vectors $\{y_1, y_2, \dots, y_n\}$ such that for every n : [5]
- $$\langle x_1, x_2, \dots, x_n \rangle = \langle y_1, y_2, \dots, y_n \rangle.$$

- Q8)** a) Let T be a triangulabel linear operator on a finite dimensional vector space. Prove that there exists an ordered orthonormal basis B of V such that the matrix of T with respect to B is upper triangular. [6]
- b) Let S be an $n \times n$ be a skew-symmetric matrix. Show that $1_n + S$ is invertible. [5]
- c) Let T be a triangulable linear operator on a finite dimensional inner product space V over F . Prove that T is normal if and only if V has an orthonormal basis consisting of eigenvectors of T . [5]



Total No. of Questions : 8]

SEAT No. :

P1229

[Total No. of Pages : 3

[5121]-14
M.A./M.Sc. (Semester - I)
MATHEMATICS
MT - 504 : Number Theory
(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) *Attempt any five questions.*
- 2) *Figures to the right indicate full marks.*

- Q1)** a) Let p denote a prime. Prove that $x^2 \equiv -1 \pmod{p}$ has a solution if and only if $p = 2$ or $p \equiv 1 \pmod{4}$. [6]
- b) Show that $61! + 1 \equiv 63! + 1 \equiv 0 \pmod{71}$ [5]
- c) Find all roots of the congruence $x^2 + x + 7 \equiv 0 \pmod{15}$. [5]
- Q2)** a) If $(a, m) = 1$, then prove that $a^{\phi(m)} \equiv 1 \pmod{m}$. [6]
- b) Solve the set of congruences $x \equiv 1 \pmod{4}$, $x \equiv 0 \pmod{3}$, $x \equiv 5 \pmod{7}$. [5]
- c) Show that 1763 is composite. [5]
- Q3)** a) Let p be an odd prime. Prove that $\left(\frac{a}{p}\right) \equiv a^{\left(\frac{p-1}{2}\right)} \pmod{p}$. [6]
- b) Prove that 3 is a quadratic residue of 13, but a quadratic nonresidue of 7. [5]
- c) Suppose that p is an odd prime. Let n denote the least positive quadratic non-residue modulo p . prove that $n < 1 + \sqrt{p}$. [5]

P.T.O.

Q4) a) If P and Q are odd and positive and $(P,Q) = 1$, then show that

$$\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right) = (-1)^{\left(\frac{P-1}{2} \times \frac{Q-1}{2}\right)} \quad [6]$$

b) If P is an odd prime, then prove that $x^2 \equiv 2 \pmod{p}$ has a solution if and only if $P \equiv 1$ or $7 \pmod{8}$. [5]

c) Find the value of $\left(\frac{-42}{61}\right)$. [5]

Q5) a) Let P denote a prime. Then prove that the largest exponent e such that

$$P^e | n! \text{ is } e = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{P^i} \right\rfloor. \quad [6]$$

b) Find all solution in positive integer $123x + 57y = 531$. [5]

c) Evaluate $\sum_{j=1}^{\infty} \mu(j!)$ [5]

Q6) a) Prove that the fields $\mathbb{Q}(\sqrt{m})$, for $m = -1, -2, -3, -7, 2, 3$ are Euclidean and so have unique factorization property. [8]

b) Show that the product of two primitive polynomials is primitive. [5]

c) Find the minimal polynomial of $1 + \sqrt{2}$. [3]

Q7) a) Let m be a negative square free rational integer. Then prove that [8]

i) The field $\mathbb{Q}(\sqrt{m})$ has units ± 1 , and these are the only units except in the cases $m = -1$ and $m = -3$.

ii) The units for $\mathbb{Q}(i)$ are ± 1 and $\pm i$

iii) The units for $\mathbb{Q}(\sqrt{-3})$ are ± 1 ,

$$\left(1 \pm \sqrt{-3} / 2\right), \left(-1 \pm \sqrt{-3} / 2\right)$$

b) Prove that the number $\beta = \sum_{j=1}^{\infty} 10^{-j!}$ is transcendental. [5]

c) Prove that 3 is prime in $\mathbb{Q}(i)$, but not a prime in $\mathbb{Q}(\sqrt{6})$. [3]

Q8) a) If $f(n) = \sum_{d|n} \mu(d)F(n/d)$ for every positive integer n , then prove that

$$F(n) = \sum_{d|n} f(d). \quad [6]$$

b) Find the highest power of 6 dividing 533! [5]

c) Find all the solutions of the congruence $9x \equiv 21 \pmod{13}$. [5]



Total No. of Questions : 8]

SEAT No. :

P1230

[Total No. of Pages : 3

[5121]-15

M.A./M.Sc. (Semester - I)

MATHEMATICS

MT - 505 : Ordinary Differential Equations

(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicate full marks.

Q1) a) Find the general solution of $y'' - 2y' = 12x - 10$. [5]

b) If $y_1(x)$ and $y_2(x)$ are two solutions of equation $y'' + P(x)y' + Q(x)y = 0$ on $[a, b]$, then prove that their wronskian $W(y_1, y_2)$ is identically equal zero or never zero on $[a, b]$. [5]

c) Verify that $y_1 = x^2$ is one solution of $x^2y'' + xy' - 4y = 0$ and find y_2 and the general solution. [6]

Q2) a) Discuss the method of undetermined coefficients to find the solution of second order differential equation with constant coefficients. [8]

b) Find the particular solution of $y'' + y = \operatorname{cosec} x$ by method of variation of parameter. [8]

Q3) a) State and prove Sturm separation theorem. [8]

b) Verify that origin is regular singular point and calculate two independent Frobenius series solution for the equation $4xy'' + 2y' + y = 0$. [8]

P.T.O.

Q4) a) Find the general solution of the system.

$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = 4x - 2y \quad [6]$$

b) Locate and classify the singular points on the x-axis.

$$x^2(x^2-1)^2 y'' - x(1-x)y' + 2y = 0. \quad [5]$$

c) Show that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any interval, and find the particular solution for which $y(0) = 2$ and $y'(0) = 3$. [5]

Q5) a) Let $u(x)$ be any non-trivial solution of $u'' + q(x)u = 0$ where $q(x) > 0$ for all $x > 0$.

If $\int_1^{\infty} q(x)dx = \infty$, then prove that $u(x)$ has infinitely many zeros on the positive x -axis. [8]

b) Find the general solution of $(1 + x^2) y'' + 2xy' - 2y = 0$ in terms of power series in x . [8]

Q6) a) Find the general solution near $x = 0$ of the hypergeometric equation $x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$. where a, b and c are constants. [8]

b) If m_1 and m_2 are roots of the auxiliary equation of the system

$$\frac{dx}{dt} = a_1x + b_1y$$

$$\frac{dy}{dt} = a_2x + b_2y$$

Which are complex conjugate but not pure imaginary, then prove that the critical point $(0,0)$ is a spiral. [8]

Q7) a) Find the general solution of $(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$ about $x = 0$ by power series method. [8]

b) Solve the following initial value problem

$$\begin{aligned} \frac{dy}{dx} &= z & y(0) &= 1 \\ \frac{dz}{dx} &= -y & z(0) &= 1. \end{aligned} \quad [8]$$

Q8) a) Let $f(x,y)$ be a continuous function that satisfies a Lipschitz condition

$$|f(x_1, y_1) - f(x_1, y_2)| \leq k|y_1 - y_2| \text{ on a strip defined by } a \leq x \leq b \text{ and } -\infty < y < \infty.$$

If (x_0, y_0) is any point of the strip, then prove that the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has one and only one solution $y = y(x)$ on the interval $a \leq x \leq b$. [10]

b) For the following system

$$\begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= -x + 2y \end{aligned} \quad [6]$$

i) Find the differential equation of path.

ii) Solve the equations to find the path.

iii) Find the critical points.



Total No. of Questions : 8]

SEAT No. :

P1231

[Total No. of Pages : 3

[5121]-21

M.A./M.Sc. (Semester - II)

MATHEMATICS

MT - 601 : General Topology

(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicate full marks.

Q1) a) Let B be a non-empty set. If B is countable, then show that there is a surjective function $f : \mathbb{Z}_+ \rightarrow B$. [4]

b) For a non-empty set X , let τ_c be a collection of all subsets U of X such that $X - U$ is either countable or all of X . Then show that τ_c is a topology on X . [6]

c) Let X be any set and let B be a basis for a topology τ on X . Prove that τ equals the collection of all unions of elements of B . [6]

Q2) a) Show that both, lower limit topology and K -topology are finer than usual topology on \mathbb{R} . [6]

b) Define projection maps and show that the collection

$S = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_1^{-1}(V) \mid V \text{ open in } Y\}$ is a subbasis for the product topology on $X \times Y$. [6]

c) Let B be a basis for a topology on a set X and $Y \subset X$, then show that $B_y = \{B \cap Y \mid B \in B\}$ forms a basis for subspace topology on Y . [4]

P.T.O.

- Q3)** a) Let Y be subspace of X and A be a subset of Y . Let \bar{A} denote the closure of A in X then show that the closure of A in Y is $\bar{A} \cap Y$. [6]
- b) If X is a Hausdorff space, then prove that a sequence of points of X converges to at most one point of X . [6]
- c) Show that every simply ordered set is a Hausdorff space in the order topology. [4]
- Q4)** a) Let X be a space satisfying T_1 axiom and A be a subset of X . Prove that the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A . [6]
- b) State and prove pasting lemma. [6]
- c) Show that if U is open in X and A is closed in X , then $U - A$ is open in X and $A - U$ is closed in X . [4]
- Q5)** a) Show that if the function $f : X \rightarrow Y$ is continuous at x , then for every convergent sequence $x_n \rightarrow x$ in X , the sequence $f(x_n)$ converges to $f(x)$. Is the converse true? Justify. [6]
- b) Let $\{X_\alpha\}$ be an indexed family of topological spaces; let $A_\alpha \subseteq X_\alpha$ for each α . If $\prod X_\alpha$ is given either product or box topology then show that, $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$. [6]
- c) Define quotient topology and given an example of a quotient map which is not an open map? [4]
- Q6)** a) Prove that the product of finitely many compact spaces is compact. [8]
- b) If A is a connected subset of a topological space X and if $A \subset B \subset \bar{A}$ then show that B is connected. [5]
- c) Give an example of connected space which is not path connected. [3]
- Q7)** a) Let X be a Hausdorff space. Prove that X is locally compact if and only if for given $x \in X$ and neighborhood U of x , there is a neighborhood V of x such that \bar{V} is compact and $\bar{V} \subseteq U$. [8]

b) Prove that compactness implies limit point compactness but not conversely. [5]

c) Let X be locally compact Hausdorff space and $A \subseteq X$. If A is closed in X or open in X , show that A is locally compact. [3]

Q8) a) State and prove Tychonoff theorem. [12]

b) State Urysohn lemma. [2]

c) Define a regular space and give its example. [2]



Total No. of Questions : 8]

SEAT No. :

P1232

[Total No. of Pages : 3

[5121]-22

M.A./M.Sc. (Semester - II)

MATHEMATICS

MT - 602 : Differential Geometry
(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicate full marks.

Q1) a) Show that for open set U in \mathbb{R}^{n+1} and vector field \bar{X} on U , there exists the maximal integral curve satisfying initial conditions. [6]

b) Find and sketch the gradient field for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x_1, x_2) = x_1^2 + x_2^3 \quad [5]$$

c) Show that gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at p . [5]

Q2) a) State and prove Lagranges multiplier theorem for smooth surfaces [6]

b) Show by example that the set of vectors tangent to a point P of a level set might be all of \mathbb{R}_p^{n+1} . [5]

c) Show that n -sphere of radius 10 is a n -surface in \mathbb{R}^{n+1} . [5]

Q3) a) Show that connected n surface in \mathbb{R}^{n+1} has exactly two normal unit vector fields. [6]

P.T.O.

- b) Show that cylinder $x_1^2 + x_2^2 = 1$ in \mathbb{R}^3 can be expressed as a level set of the function $f(x_1, x_2, x_3) = \frac{-x_1^2}{4} - \frac{x_2^2}{4} + \cos(x_1^2 + x_2^2)$. [5]
- c) Show that the parametrized curve $\alpha(t) = (\cos(at + b), \sin(at + b), t + b)$ is geodesic in the cylinder $x_1^2 + x_2^2 = 1$ in \mathbb{R}^3 . [5]
- Q4)** a) Show that for a compact connected oriented \cap -Surface S in \mathbb{R}^{n+1} , the Gauss map maps S onto the unit sphere S^n [8]
- b) Show that if $\alpha: I \rightarrow \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$. [4]
- c) Find the velocity and acceleration for the parametrized curve $\alpha(t) = (\cos t, \sin t, t)$ [4]
- Q5)** a) Write a note on Levi-Civita parallelism [6]
- b) Define parallel transport. Show that it is a one one, onto, linear map from S_p to S_q . [5]
- c) Show that weingarten map L_p is a Selfadjoint map. [5]
- Q6)** a) Compute $\nabla v f$ where $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is defined by $f(x_1, x_2, x_3) = x_1 x_2 x_3^2, \bar{v} = (1, 1, 1, a, b, c)$ where $n = 2, \bar{v} \in \mathbb{R}_p^3, p \in \mathbb{R}^3$ [6]
- b) Compute the curvature of sphere in \mathbb{R}^3 oriented by outward normal [5]
- c) On each compact oriented n -surface S in \mathbb{R}^{n+1} there exists a point P such that the second fundamental form at P is definite [5]
- Q7)** a) Find the Gaussian curvature of the ellipsoid in \mathbb{R}^3 . [6]
- b) Compute the Principal curvatures of the hyperboloid $-x_1^2 + x_2^2 + x_3^2 = 1$ in \mathbb{R}^3 oriented by inward normal. [5]
- c) Write the note on the quadratic form associated with self adjoint linear transformation [5]

- Q8)** a) Describe the spherical image for the paraboloid $-x_1 + x_2^2 + \dots + x_{n+1}^2 = 0$ oriented by outward normal. **[6]**
- b) Show that mobius band is unorientable 2-surface **[5]**
- c) Find the integral curve through $P = (1,1)$ of vector field $X(p) = -P$ on \mathbb{R}^2 . **[5]**



Total No. of Questions : 8]

SEAT No. :

P1233

[Total No. of Pages : 3

[5121]-23

M.A./M.Sc. (Semester - II)

MATHEMATICS

MT-603: Groups and Rings

(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) *Attempt any five questions.*
- 2) *Figures to the right indicate full marks.*

Q1) a) Prove that set of 2×2 matrices with determinant 1 having entries from \mathbb{Q} forms a group under matrix multiplication. Is this group abelian? Justify. [5]

b) Find the inverse of the element $A = \begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in $GL(2, \mathbb{Z}_{11})$. [5]

c) Prove that a cyclic group is isomorphic to \mathbb{Z} or \mathbb{Z}_n , for some $n \in \mathbb{N}$. [6]

Q2) a) Prove that $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are not isomorphic, Also, prove that group of quaternions Q_8 is not isomorphic to the dihedral group D_4 with 8 elements. [5]

b) Prove that all the groups of odd order having atmost 19 elements are abelian. [5]

c) Suppose that H is a proper subgroup of \mathbb{Z} under addition and H contains 25, 10 and 15. Determine H. [6]

P.T.O.

- Q3)** a) Let G be a group and $Z(G)$ be the center of G . Prove that, if $G/Z(G)$ is cyclic, then G is abelian. [5]
- b) Prove that subgroup of a cyclic group is cyclic. [5]
- c) Prove that $SL(2, \mathbb{R})$ is a normal subgroup of $GL(2, \mathbb{R})$. [6]
- Q4)** a) Find the inverse and the order of each of the following permutations in S_{16} [5]
- i) $(16\ 2\ 4\ 9)(14\ 3\ 1\ 7)(6\ 5)$
- ii) $(3\ 15\ 14)(1\ 2\ 6)(13\ 7\ 12)$.
- b) If N is a normal subgroup of a group G and $|G/N| = m$, show that $x^m \in N$ for all x in G . [5]
- c) State and prove the Lagrange's theorem for finite groups. Is the converse of the theorem true? Justify. [6]
- Q5)** a) State and prove the first isomorphism theorem. [5]
- b) Let $\phi: \mathbb{C}^* \rightarrow \mathbb{C}^*$ be defined as $\phi(x) = x^4$. Prove that ϕ is a homomorphism. Also find its kernel and state whether ϕ is an automorphism. [5]
- c) If $\tau = (7\ 1\ 4)(5\ 2)$, $\rho = (8\ 4\ 6\ 9\ 1)(10\ 2\ 11) \in S_{11}$. Then find $\tau^{-1}\rho\tau$ and $\rho^{-1}\tau\rho$. [6]
- Q6)** a) Let G be a finite abelian group and let a be an element of maximum order in G . Show that for any element b in G , $|b|$ divides $|a|$. [5]
- b) Determine all the homomorphisms from \mathbb{Z}_9 to \mathbb{Z}_{27} . [5]
- c) Find all the non isomorphic abelian groups of order 5400. [6]

- Q7)** a) Determine all the groups of order 6. [5]
- b) Characterize those integers n such that the only abelian groups of order n are cyclic. [5]
- c) Let G be a finite group and let p be a prime. If p^k divides $|G|$, then prove that G has at least one subgroup of order p^k . [6]
- Q8)** a) Let $G = \mathbb{Z} \oplus \mathbb{Z}$ and $H = \{(x, y) | x, y \text{ are even integers}\}$. Show that H is a subgroup of G . Determine the order of G/H to which familiar group is G/H ? [5]
- b) Prove that every group of prime power order has non trivial center. [5]
- c) Prove that the groups of order 21 and 22 are not simple. [6]



[5121]-24

M.A./M.Sc. (Semester - II)

MATHEMATICS

MT - 604: Complex Analysis
(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicate full marks.

Q1) a) Let f and g be analytic on G and Ω respectively and suppose $f(G) \subset \Omega$. Prove that $g \circ f$ is analytic on G and $(g \circ f)'(z) = g'(f(z))f'(z)$ for all z in G . [8]

b) If $\sum_{n=0}^{\infty} a_n(z-a)^n$ is a given power series with radius of convergence R then prove that $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$ if this limit exists. [6]

c) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a^n z^n$, $a \in \mathbb{C}$. [2]

Q2) a) Define a branch of logarithm. Let f be a branch of a logarithm on an open connected subset G on \mathbb{C} . Show that the totality of branches of $\log z$ are functions $f(z) + 2\pi ik$, $k \in \mathbb{Z}$. [8]

b) Let $f: G \rightarrow \mathbb{C}$ is analytic and that G is connected. Show that if $f(z)$ is real for all z in G then f is constant. [4]

c) For the point $z = 3 + 2i$ give the corresponding point of the $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. [4]

P.T.O.

- Q3)** a) Define Mobius transformation and prove that every Mobius transformation maps circles of \mathbb{C}_∞ onto circles of \mathbb{C}_∞ . [6]
- b) Let G be either the whole plane \mathbb{C} or some open disk. if $u : G \rightarrow \mathbb{R}$ is a harmonic function then prove that u has a harmonic conjugate. [5]
- c) Find a fixed points of a dilation, a translation and the inversion on \mathbb{C}_∞ . [5]

- Q4)** a) Let $f : G \rightarrow \mathbb{C}$ be analytic and suppose $\bar{B}(ajr) \subset G (r > 0)$. If $\gamma(t) = a + re^{it}, 0 \leq t \leq 2\pi$ then prove that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw \text{ for } |z-a| < r. \quad [8]$$

- b) State and prove Cauchy's Estimate. [4]
- c) Let $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$. Then find $\int_{\gamma} z^n dz$ for every integer n . [4]

- Q5)** a) State and prove Liouville's Theorem. [6]

- b) If $\gamma : [0,1] \rightarrow \mathbb{C}$ is a closed rectifiable curve and $a \in \{\gamma\}$ then prove that $\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$ is an integer. [5]

- c) Find all entire functions f such that $f(x) = e^x$ for all x in \mathbb{R} . [5]

- Q6)** a) Let G be an open subset of the plane and $f : G \rightarrow \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $\eta(\gamma, w) = 0$ for all w in $\mathbb{C} - G$ then prove that for ' a ' in $G - \{\gamma\}$

$$\eta(\gamma, a) f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz. \quad [6]$$

- b) State and prove Morera's theorem. [5]
- c) Let f be analytic in $B(a, R)$ and suppose that $f(a) = 0$. Show that a is a zero of multiplicity M iff $f^{(m-1)}(a) = \dots = f'(a) = 0, f^{(m)}(a) \neq 0$. [5]

Q7) a) If f has an isolated singularity at a then prove that the point $z = a$ is a removable singularity if and only if $\lim_{z \rightarrow a} (z - a)f(z) = 0$ [8]

b) Show $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ [6]

c) State the Cauchy-Residue theorem. [2]

Q8) a) State and prove Rouché's theorem. [6]

b) Let G be a region in \mathbb{C} and f an analytic function on G . Suppose there is a constant M such that $\limsup_{z \rightarrow a} |f(z)| \leq M$ for all a in $\partial_{\infty} G$. Prove that $|f(z)| \leq M$ for all z in G . [5]

c) Does there exist an analytic function $f : D \rightarrow D$ where $D = \{z : |z| < 1\}$ with $f\left(\frac{1}{2}\right) = \frac{3}{4}$ and $f'\left(\frac{1}{2}\right) = \frac{2}{3}$? [5]



Total No. of Questions : 8]

SEAT No. :

P1235

[Total No. of Pages : 3

[5121]-25

M.A./M.Sc. (Semester - II)

MATHEMATICS

MT-605: Partial Differential Equation
(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicates full marks.

Q1) a) Prove that the Pffafian differential equation : [6]

$$\bar{X}.d\bar{r} = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0$$

is integrable if and only if $\bar{X}.\text{Curl}\bar{X} = 0$

b) Find the general integral of $z(xp - yq) = y^2 - x^2$. [5]

c) Find the complete integral of $z^2(p^2z^2 + q^2) = 1$ by Charpit's Method. [5]

Q2) a) Explain the method of solving the following first order partial differential equations:

i) $f(z, p, q) = 0$. [3]

ii) $g(x, p) = h(y, q)$. [3]

b) Show that following equations are compatible and find one parameter family of their common solutions $f = xp - yq - x = 0$, $g = x^2p + q - xz = 0$. [5]

c) Solve the differential equation $z^2(p^2z^2 + q^2) = 1$ [5]

P.T.O.

Q3) a) Show that the general solution of the Quasi-linear equation

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z),$$

where P, Q, R are continuously differentiable functions of x, y, z is $F(u, v) = 0$, where F is an arbitrary differentiable function of u, v and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solutions of the system.

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)}. \quad [8]$$

b) Find the characteristic strip of the equation $pq = ny$ and obtain the equation of integral surface through the curve : $z = x, y = 0$. [8]

Q4) a) Reduce the equation

$$y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y$$

in to cononical form and solve it. [8]

b) Find the complete integral of the equation $f = p^2 x + q^2 y - z = 0$ by Jacobi's method. [8]

Q5) a) State Dirichlet's problem for rectangle and solve it. [8]

b) Using D'Alemberts solution of infinite string find the solution of

$$y_{tt} = c^2 y_{xx}, 0 < x < \infty, t > 0,$$

$$y(x, 0) = u(x), y_t(x, 0) = v(x), x \geq 0,$$

$$y(0, t) = 0, t \geq 0. \quad [8]$$

Q6) a) State and prove Harnack's theorem. [6]

b) Find the characteristic lines of the hyperbolic equation

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0. \quad [5]$$

c) Suppose that $u(x, y)$ is harmonic function in a bounded domain D and continuous in $\bar{D} = D \cup B$ then prove that u attains its maximum on the boundary B of D. [5]

Q7) a) State and prove Kelvin's inversion theorem. **[8]**

b) Solve the boundary value problem

$$u_t = u_{xx}, 0 < x < l, t > 0,$$

$$u(0,t) = u(l,t) = 0, t \geq 0,$$

$$u(x,0) = x(l-x), 0 \leq x \leq l. \quad \text{[8]}$$

Q8) a) Prove that the Neumann problem has unique solution upto addition of a constant. **[6]**

b) Solve the following diffusion equation using Fourier transform technique:

$$u_t = ku_{xx}, -\infty < x < \infty, t > 0, \quad \text{[6]}$$

$$u(x,0) = F(x), -\infty < x < \infty$$

c) Is the surface $x^2 + y^2 + z^2 = cx^{\frac{2}{3}}$ equipotential? If yes, find the potential function. **[4]**



Total No. of Questions : 4]

SEAT No. :

P1236

[Total No. of Pages : 2

[5121]-26

M.A./M.Sc. (Semester - II)

MATHEMATICS

**MT-606: Object Oriented Programming using C++
(2008 Pattern)**

Time : 2 Hours]

[Max. Marks : 50

Instructions to the candidates:

- 1) *Question One is compulsory.*
- 2) *Attempt any two questions from Q.2, Q.3 and Q.4.*
- 3) *Figures to the right indicate full marks.*

Q1) Attempt any Ten of the following :

[20]

- a) Write syntax of output stream and input stream.
- b) Write a short note on function prototype.
- c) What are disadvantages of macros?
- d) What are C++ keywords? Give four examples.
- e) What is data encapsulation?
- f) Write output of following program.

```
# include < iostream.h >
int main ( )
{
cout << "Mathematics is queen of science",
return 0;
}
```
- g) Define friend function.
- h) What do you mean by hybrid inheritance?
- i) Write a function to read a matrix of size 5×6 from the keyboard using "for" loop.

P.T.O.

- j) State one difference between break and continue.
 - k) What is operator overloading?
 - l) Explain the term “message passing”.
- Q2)** a) Write a program in C++ to find Euclidean distance between two points in XY plane with output. [5]
- b) Write a note on inline functions. [5]
- c) Compare dynamic memory management in C and C++. [5]
- Q3)** a) Write a note general form of class declaration. [5]
- b) Write a program in C++ to find simple interest. [5]
- c) What is difference between constructor and destructor? [5]
- Q4)** a) Define : [9]
- i) Call by value
 - ii) Call by reference
 - iii) Return by reference with examples.
- b) Write benefits of OOP (Object Oriented Programming). [6]



Total No. of Questions : 8]

SEAT No. :

P1237

[Total No. of Pages : 2

[5121]-31

M.A./M.Sc. (Semester - III)

MATHEMATICS

MT-701: Functional Analysis
(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicate full marks.

- Q1)** a) Show that quotient space of a Banach Space is a Banach Space. [6]
b) Show that a normed linear space N is a Banach Space iff $\{x \in N / \|x\| = 1\}$ is complete. [5]
c) Show that set of all continuous linear transformations from Banach Space to itself is a Banach Space. [5]
- Q2)** a) State and prove Hahn-Banach theorem for Banach Spaces. [6]
b) Show that if M is a closed linear subspace of Normed linear space N and $x_0 \in N, x_0 \notin M$, then \exists a functional f_0 in N^* . Such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$. [5]
c) State and prove Open Mapping theorem for Banach spaces. [5]
- Q3)** a) State and prove Banch-Steinhaus theorem for Banach spaces. [6]
b) Show that closed convex subset C of a Hilbert space H contains a unique vector of smallest norm. [5]
c) Show that for closed linear subspace M of the Hilbert space $M = M^{\perp\perp}$. [5]

P.T.O.

- Q4)** a) State and prove closed graph theorem for Banach spaces. [6]
- b) Show that if M is a closed linear subspace of Hilbert space H , then $H = M \oplus M^\perp$. [5]
- c) Show that every nonzero Hilbert space contains a complete orthonormal set. [5]
- Q5)** a) Prove that product of self adjoint operators is self adjoint iff they commute. [6]
- b) State and prove Bessels inequality for Hilbert spaces. [5]
- c) Show that unitary operators on the Hilbert space forms group under composition. [5]
- Q6)** a) Show that adjoint operator $T \rightarrow T^*$ on $B(H)$ has following properties. [6]
- i) $(T_1 T_2)^* = T_2^* T_1^*$
- ii) $\|T^*\| = \|T\|$
- iii) $(T_1 + T_2)^* = T_1^* + T_2^*$
- b) State and prove Schwarz inequality for vectors in Hilbert space. [5]
- c) Show that an operator T on H is unitary iff it is an isometric isomorphism of H onto itself. [5]
- Q7)** a) Show that if T is normal operator then each eigenspace M_i reduces T . [6]
- b) State and prove spectral theorem for Hilbert spaces. [10]
- Q8)** a) Show that an operator on Hilbert space is normal iff its real and imaginary part commutes. [6]
- b) Show that if H is a Hilbert space and f is arbitrary functional on H^* , then there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H . [10]



Total No. of Questions : 8]

SEAT No. :

P1238

[Total No. of Pages : 3

[5121]-32

M.A./M.Sc. (Semester - III)

MATHEMATICS

MT-702: Ring Theory

(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to right indicate full marks.
- 3) All questions carry equal marks.

Q1) a) Prove that only finite integral domain is a field. [6]

b) Let R be a ring of all continuous functions from the closed interval $[0, 1]$ to R . [5]

Show that the function f in R , defined by, $f(x) = x^{-1/2}$ is neither a unit nor a zero divisor in R .

Is the function g , defined by,

$$g(x) = 0 \quad \text{if } 0 \leq x \leq \frac{1}{2}$$

$$= x^{-1/2} \quad \text{if } \frac{1}{2} \leq x \leq 1 \text{ a zero divisor in } R?$$

c) Let R be a ring with identity and let S be a subring of R containing the identity. Prove that if u is a unit in S then u is a unit in R . Show by an example that the converse is not true. [5]

Q2) a) Define a Boolean ring and prove that the only Boolean ring is an integral domain in Z_2 . [5]

b) If R is an integral domain and $p(x), q(x) \in R(x)$, then prove that [6]

i) $\deg (p(x) q(x)) = \deg p(x) + \deg q(x)$

ii) units of $R(x)$ are just the units of R

iii) $R(x)$ is an integral domain.

P.T.O.

c) Define the map $\phi : \mathbb{Q}[x] \rightarrow \mathbb{Q}$ by

$$f(x) \rightarrow f(0)$$

Show that the above map is a ring homomorphism. Find the kernel of this homomorphism. Is it onto? [5]

Q3) a) If $\phi : \mathbb{R} \rightarrow \mathbb{S}$ is a ring homomorphism then prove that $\ker \phi$ is an ideal of \mathbb{R} and image of ϕ is a subring of \mathbb{S} . [6]

b) Show that the ideal $I = (z, x)$ in $\mathbb{Z}[x]$ is not a principal ideal. [5]

c) Let \mathbb{R} be the ring of all functions from $[0, 1]$ to set of all real numbers \mathbb{R} and M_a be defined by [5]

$$M_a = \{f \in \mathbb{R} \mid f(a) = 0, a \in [0, 1]\}$$

Prove that M_a is a maximal ideal of \mathbb{R} .

Q4) a) Prove that every ideal in a Euclidean domain is principal. [6]

b) Prove that the ring $\mathbb{R} = \mathbb{Z}[\sqrt{-5}]$ is not an Euclidean domain. [5]

c) Show that every non-zero prime ideal in a principal ideal domain is a maximal ideal. [5]

Q5) a) Define Dedekind-Hasse norm. If the integral domain \mathbb{R} has a Dedekind-Hasse norm then prove that \mathbb{R} is a principal ideal domain. [6]

b) Prove that the ring $\mathbb{R} = \mathbb{Z}[2i] = \{a + 2bi \mid a, b \in \mathbb{Z}\}$ is not U.F.D. [5]

c) Prove that the quotient ring $\frac{\mathbb{Z}(i)}{(1+i)}$ is a field of order \mathbb{Z} . [5]

Q6) a) If I is an ideal of the ring \mathbb{R} and $(I) = I[x]$ is an ideal of $\mathbb{R}[x]$ generated by I then prove that [6]

$$\frac{\mathbb{R}[x]}{(I)} \cong \left(\frac{\mathbb{R}}{I} \right)[x].$$

- b) Prove that the ideals (x) and (x, y) are prime ideals in $\mathbb{Q}[x, y]$. [5]
 Also prove that (x, y) is maximal but (x) is not maximal.
- c) If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ be a polynomial of degree n with integer coefficient. If $\frac{r}{s} \in \mathbb{Q}$ is in lowest term and $\frac{r}{s}$ is a root of $p(x)$. Then show that $r|a_0$ and $s|a_n$. [5]

- Q7)** a) State and prove Eisenstein's criterion for irreducibility of a polynomial. [6]
 b) Find all monic irreducible polynomials of degree ≤ 3 in $\mathbb{F}_2[x]$. [5]
 c) Construct the field with 9 elements. [5]

- Q8)** a) If R is a PID then show that there exist a multiplicative Dedekind-Hasse norm on R . [8]
 b) Define the Noetherian ring and show that the quotient of Noetherian ring is Noetherian. [5]
 c) i) State only Hilbert basis theorem. Is the converse true?
 ii) Give an example of ring which is not Noetherian. [3]



Total No. of Questions : 8]

SEAT No. :

P1239

[Total No. of Pages : 3

[5121]-33

M.A./M.Sc. (Semester - III)

MATHEMATICS

MT-703: Mechanics

(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicate full marks.

Q1) a) Explain the principle of virtual work and derive D' Alembert's equation of motion. [8]

b) Find the force generated by the potential $\phi(x, y, z) = -x^2y - 2z + g$. Is this force conservative? Justify your answer. [4]

c) Explain the concept of degrees of freedom and find the degrees of freedom for a canonical pendulum. [4]

Q2) a) Derive Hamilton's equations of motion for simple pendulum. [6]

b) Is $x\dot{x} + y\dot{y} + \dot{x}y + x\dot{y} = \text{constant}$, a holonomic constraint? Justify your answer. [5]

c) Show that the Lagrange's equation of motion can also be written as

$$\frac{\partial L}{\partial t} - \frac{d}{dt} \left(L - \sum \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) = 0 \quad [5]$$

Q3) a) Show that the curve is a catenary for which the area of surface of revolution is maximum when revolved about y-axis. [8]

P.T.O.

- b) Find the Hamilton's canonical equations for a one-dimensional harmonic oscillator for which the kinetic energy is $\frac{1}{2}m\dot{x}^2$ and the potential energy is $\frac{1}{2}kx^2$. [4]
- c) Consider motion of a free particle having mass m in a plane. Express its kinetic energy in terms of plane polar co-ordinates and their time derivatives. [4]
- Q4)** a) Write a note on Brachistochrone problem. [6]
- b) Deduce Newton's second law of motion from Hamilton's principle. [7]
- c) Explain the conservative and scleronomic system. [3]
- Q5)** a) A planet moves under the inverse square law of attractive force. Find Lagrangian L , Hamiltonian H , and Routhian R for the planet. [8]
- b) Write Hamilton's equation using poisson brackets. Show that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$, where H denotes Hamiltonian. [4]
- c) Prove that $[f, gh] = g[f, h] + h[f, g]$. [4]
- Q6)** a) State and prove rotation formula. [6]
- b) Define infinitesimal rotations. Show that the matrix representing infinitesimal rotations is antisymmetric. [6]
- c) Explain the active and passive view of co-ordinates transformations. [4]
- Q7)** a) State and prove Euler's theorem about general displacement of a rigid body. [6]

b) Derive the matrix of transformation in terms of the Euler angles : (ϕ, θ, ψ) . [6]

c) Show that the following transformation is canonical $Q = \log(1 + \sqrt{q} \cos p)$
 $P = 2(1 + \sqrt{q} \cos p)\sqrt{q} \sin p$. [4]

Q8) a) Derive the following differential equation for the path of the particle in the central force field. [6]

$$\frac{d^2u}{d\theta^2} + u = -\frac{f\left(\frac{1}{u}\right)}{mh^2u^2}.$$

b) Define central force motion. Show that it is always planar. Further show that the areal velocity is constant. [5]

c) State and prove Kepler's second law of planetary motion. [5]



Total No. of Questions : 8]

SEAT No. :

P1240

[Total No. of Pages : 3

[5121]-34

M.A./M.Sc. (Semester - III)

MATHEMATICS

MT-704 : Measure and Integration

(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicate full marks.
- 3) All symbols have their usual meanings.

Q1) a) Define following terms with suitable example. [6]

- i) Outer Measure
- ii) Signed measure
- iii) Hausdorff Measure

b) Let $\{E_i\}$ be a sequence of measurable sets. If $E_1 \subseteq E_2 \subseteq \dots$, then show that $\mu(\lim E_i) = \lim \mu(E_i)$. [5]

c) Let (X, \mathcal{B}, μ) be a measure space and $Y \in \mathcal{B}$. Let \mathcal{B}_Y consist of those sets of \mathcal{B} that are contained in Y and $\mu_Y(E) = \mu(E)$ if $E \in \mathcal{B}_Y$. Then show that $(Y, \mathcal{B}_Y, \mu_Y)$ is a measure space. [5]

Q2) a) Define a σ -algebra. Show that the class of measurable sets \mathcal{M} is a σ -algebra. [6]

b) Let c be any real number and let f and g be real valued measurable functions defined on the same measurable set E . Show that $f + c, cf, f + g, f - g$ and fg are measurable. [6]

c) Define Hausdorff measure and show that it is invariant under translation. [4]

Q3) a) Let ν be a signed measure on the measurable space (X, \mathcal{B}) then prove that there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \phi$. [6]

b) State and prove Fatou's Lemma. [6]

c) Give an example to show that the Hahn decomposition need not be unique. [4]

P.T.O.

- Q4) a)** If μ is a measure on a ring \mathcal{R} and the set function μ^* is defined on $\mathcal{H}(\mathcal{R})$ by $\mu^*(E) = \inf \left[\sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathcal{R}, n = 1, 2, \dots, E \subset \bigcup_{n=1}^{\infty} E_n \right]$ Then show that (i) for $E \in \mathcal{R}, \mu^*(E) = \mu(E)$, (ii) μ^* is an outer measure on $\mathcal{H}(\mathcal{R})$. **[8]**
- b) If μ is a σ -finite measure on a ring \mathcal{R} then show that it has unique extension to the σ -ring $\mathcal{N}(\mathcal{R})$. **[8]**
- Q5) a)** Give an example of measure μ and ν , on the measurable space, such that non of the relation $\mu \ll \nu, \nu \ll \mu$ and $\mu \perp \nu$ holds. **[6]**
- b) i) Define signed measure **[6]**
- ii) Let ν be a signed measure on (X, \mathcal{B}) and $E \in \mathcal{B}$ with $\nu(E) > 0$. Then show that there exist A , a set positive with respect to ν , such that $A \subset E$ and $\nu(A) > 0$.
- c) Show that $\mu(E_1 \Delta E_2) = 0$ implies $\mu E_1 = \mu E_2$ provided that $E_1, E_2 \in \mathcal{B}$, **[4]**
- Q6) a)** Define product measure. Let E (subset of $X \times Y$) a set in $\mathcal{R}_{\sigma\delta}$ and x be a point of X . Then show that E_x (x cross section E) is a measurable subset of Y . **[6]**
- b) Let F be a bounded linear functional on $L^p(\mu)$ with $1 \leq p < \infty$ and μ a σ -finite measure. Then show that there is a unique element g in L^q where $1/p + 1/q = 1$. such that $F(f) = \int fg d\mu$ with $\|F\| = \|g\|_q$. **[6]**
- c) Show that the cantor set is a compact, non empty perfect and totally disconnected metric space. **[4]**
- Q7) a)** Let E be a set in $\mathcal{R}_{\sigma\delta}$ with $\mu \times \nu(E) < \infty$. Then show that the function g defined by $g(x) = \mu E_x$ is a measurable function of x and $\int g d\mu = \mu \times \nu(E)$ **[6]**

b) Let $\mu(X)$ be a finite. If $\phi(f) = \int \frac{|f|}{1+|f|} d\mu$ and $\rho(f,g) = \phi(f-g)$, where f, g are measurable and finite valued a.e. Then.

i) Show that ρ is a metric space [2]

ii) Show that convergence in this metric is equivalent to convergence in measure. [2]

iii) Show that the metric space with metric ρ is complete. [2]

c) Let E and F are measurable sets, $f \in L(E, \mu)$ and $\mu(E \Delta F)$, then show that $f \in L(F, \mu)$ and $\int_E f d\mu = \int_F f d\mu$. [4]

Q8) a) Let μ be a measure on an algebra \mathcal{G} and μ^* the outer measure induced by μ . Then prove that the restriction $\bar{\mu}$ of μ^* to the μ^* -measurable sets is an extension of μ to σ -algebra containing \mathcal{G} . [8]

b) If μ^* is a Carathéodory outer measure with respect to \mathcal{F} then prove that every function in \mathcal{F} is μ^* -measurable. [8]



Total No. of Questions : 8]

SEAT No. :

P1241

[Total No. of Pages : 3

[5121]-35

M.A./M.Sc. (Semester - III)

MATHEMATICS

MT-705 : Graph Theory

(2008 Pattern)

Time : 3 Hour]

[Max. Marks : 80

Instructions to the candidates:

- 1) *Attempt any five questions.*
- 2) *Figures to the right indicate full marks.*

Q1) a) Show that every set of six people contains at least three mutual acquaintances or three mutual strangers. [6]

b) Prove that every graph with n vertices and k edges has at least $n-k$ components. [4]

c) Prove that an edge is a cut edge if and only if it belongs to no cycle. [6]

Q2) a) Find the number of simple graphs with a vertex set X of size n . [5]

Draw all the non isomorphic simple graphs on a fixed set of four vertices.

b) Prove that the Petersen graph has ten 6-cycles. [5]

c) Prove that a graph is bipartite if it has no odd cycle. [6]

Q3) a) Prove that for a connected nontrivial graph with exactly $2k$ odd vertices, the minimum number of trails that decompose it is $\max\{k, 1\}$. [8]

b) Show that if G is a simple n - vertex graph with $\delta(G) \geq \frac{(n-1)}{2}$, then G is connected [4]

c) Show that the number of vertices in a self-complementary graph is either $4k$ or $4k + 1$, where k is a positive integer. [4]

P.T.O.

Q4) a) Prove that if T is a tree with k edges and G is a simple graph with $\delta(G) \geq k$ then T is a subgraph of G . [7]

b) Show that every graph has an even number of vertices of odd degree. [3]

c) Show that there exists a simple graph with 12 vertices and 28 edges such that the degree of each vertex is either 3 or 5. Draw this graph. [6]

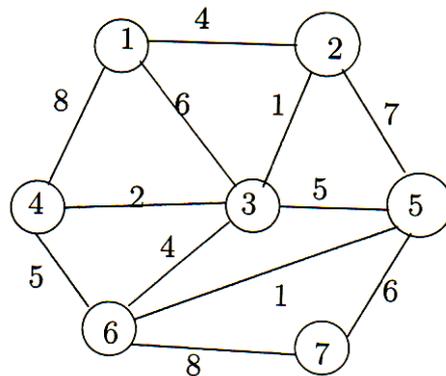
Q5) a) Prove that the center of a tree is a vertex or an edge. [7]

b) Let T be a tree with average degree a . Determine $n(T)$ in terms of a . [3]

c) Prove that if G is a simple graph with $\text{diam } G \geq 3$, then $\text{diam } \bar{G} \leq 3$. [6]

Q6) a) Prove that if G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G . [10]

b) Use Dijkstra's algorithm to find shortest distance from a vertex 1 to any other vertex in the following graph [6]



Q7) a) Prove that in a connected weighted graph G , Kruskal's Algorithm constructs a minimum-weight spanning tree. [6]

b) Determine whether the sequence $(5, 5, 4, 3, 2, 2, 2, 1)$ is graphic. Provide a construction or a proof of impossibility. [4]

c) Show that the connectivity of the hypercube Q_k is k . [6]

- Q8)** a) Prove that every component of the symmetric difference of two matchings is a path or an even cycle. [5]
- b) Define clique number and independence number of a graph G with an example. Prove that for every graph G , $\chi(G) \geq \omega(G)$ and where $\omega(G)$ is the clique number of G and $\alpha(G)$ is the independence number of G . [5]
- c) Prove that if G is a connected graph, then an edge cut F is a bond if and only if $G - F$ has exactly two components. [6]



$$\chi(G) \geq \frac{n(G)}{\alpha(G)},$$

Total No. of Questions : 8]

SEAT No. :

P1242

[Total No. of Pages : 2

[5121]-41
M.A./M.Sc.
MATHEMATICS
MT - 801 : Field Theory
(2008 Pattern) (Semester - IV)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) *Attempt any five questions of the following.*
- 2) *Figures to the right indicate full marks.*

Q1) a) Let $p(x)$ be an irreducible polynomial in $F[x]$ and 'u' be a root of $f(x)$ in an extension E of F . Then prove that $F(u)$, the subfield of E generated by F and u is the set

$$\{b_0 + b_1 u + \dots + b_m u^m / b_0 + b_1 x + \dots + b_m x^m \in F[x]\}$$

Further show that if degree of $p(x)$ is 'n' then $\{1, u, \dots, u^{n-1}\}$ forms a basis of $F(u)$ over F . **[8]**

- b) Define normal extension and illustrate it by an example. **[5]**
- c) Is $f(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$ irreducible over \mathbb{Z}_2 . **[3]**

Q2) a) If K is a splitting field of $f(x) \in F[x]$ over F then show that K is an algebraic extension of F . **[8]**

- b) If $p(x) = x^2 - x - 1 \in \mathbb{Z}_3[x]$, then show that there exist an extension K to \mathbb{Z}_3 with nine elements having all roots of $p(x)$. **[8]**

Q3) a) Show that doubling the cube and squaring the circle are impossible by using ruler and compass. **[8]**

- b) Determine the minimal polynomial over \mathbb{Q} for the element $\sqrt{-1 + \sqrt{2}}$. **[5]**
- c) Construct a field with 4-elements. **[3]**

Q4) a) If E is a Galois extension of F and K is any subfield of E containing F then with usual notation prove that $K = E_{G(E/K)}$. **[8]**

- b) Let E be an extension of a field F . Define the group of F -automorphisms of E with an example. **[5]**
- c) Examine whether the polynomial $x^4 + x + 1 \in \mathbb{Q}(x)$ is a separable polynomial. **[3]**

P.T.O.

- Q5)** a) If K is a field of characteristic $p \neq 0$ and if K is a perfect field then show that $K^p = K$. [8]
- b) If $[F(\alpha) : F]$ is odd then prove that $F(\alpha) = F(\alpha^2)$. [4]
- c) Prove that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{2} + \sqrt{3}$ are algebraic over \mathbb{Q} . [4]
- Q6)** a) Determine the splitting field and its degree over \mathbb{Q} for the polynomial $f(x) = x^4 - 2$. [8]
- b) Let E be the splitting field of $x^n - a \in F[x]$, then prove that $G(E/F)$ is solvable group. [8]
- Q7)** a) Show that the group $G = G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ of \mathbb{Q} -automorphisms of $\mathbb{Q}(\sqrt[3]{2})$ is trivial. [8]
- b) State the fundamental theorem of Galois theory. [4]
- c) Let $F = \mathbb{Q}(\sqrt{2})$ and $E = \mathbb{Q}(\sqrt[4]{2})$. Show that E is a normal extension of F and F is a normal extension of \mathbb{Q} but E is not a normal extension of \mathbb{Q} . [4]
- Q8)** a) Show that a finite field F of p^n elements has exactly one subfield with p^m elements for each divisor m of n . [6]
- b) Determine Galois group of $f(x) = x^3 - x + 1 \in \mathbb{Q}[x]$ over \mathbb{Q} . [5]
- c) Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. [5]



Total No. of Questions : 8]

SEAT No. :

P1243

[Total No. of Pages : 3

[5121]-42

M.A./M.Sc. (Semester - IV)

MATHEMATICS

MT - 802 : Combinatorics

(2008 Pattern)

Time : 3 Hour]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicate full marks.

Q1) a) How many integers between 1000 and 10000 are there (Leading zeros not allowed) with [6]

- i) Repetition of digits allowed but with no 2 or 4?
- ii) Distinct digits and at least one of 2 and 4 must appear?

b) Prove by combinatorial argument that [6]

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \dots + \binom{n}{r} = \binom{n+1}{r+1}$$

Hence, evaluate the sum

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \dots + (n-2)(n-1)n.$$

c) Find all derangements of 1,2,3,4,5 with the help of associated chessboard of darkened squares. [4]

Q2) a) What fraction of all arrangements of INSTRUCTOR have three consecutive vowels? [6]

b) How many non negative integer solutions are there to the inequalities $x_1 + x_2 + \dots + x_6 \leq 20$ and $x_1 + x_2 + x_3 \leq 7$? [6]

c) Among 40 toy robots, 28 have a broken wheel or are rusted but not both, 6 are not defective, and the number with a broken wheel equals the number with rust. Find how many robots are rusted? [4]

P.T.O.

- Q3)** a) How many ways are there to select 300 chocolate candies from seven types if each type comes in boxes of 20 and if at least one but not more than five boxes of each type are chosen? [6]
- b) Find ordinary generating function whose coefficient a_r equals r . Hence evaluate the sum $0 + 1 + 2 + 3 + \dots + n$. [6]
- c) How many ways are there to distribute 15 identical objects into four different boxes if the number of objects in box 4 must be a multiple of 3? [4]

- Q4)** a) Use generating functions to find the number of ways to select 10 balls from a large pile of red, white and blue balls if the selection has at most two red balls. [6]
- b) How many permutations of the 26 letters are there that contain none of the sequences MATH, RUNS, FROM or JOE? [6]
- c) Find generating function for modeling the number of 5-combinations of the letters M,A,T,H. in which M and A can appear any number of times but T and H appear at most once. Which coefficient in this generating function do we want? [4]

- Q5)** a) How many r digit quaternary sequences are there in which the total number of 0's and 1's is even? [4]
- b) Using inclusion exclusion principle, find the number of ways to distribute 25 identical balls into 6 distinct boxes with at most 6 balls in any of the first three boxes [6]
- c) Solve the recurrence relation. [6]

$$a_n = 2 a_{\frac{n}{2}} + 2, \quad n \geq 4, \quad \text{with } a_2 = 1$$

(assume that n is power of 2)

Q6) a) Using generating functions, solve the recurrence relation. [6]

$$a_n = a_{n-1} + n(n-1); \quad a_0 = 1$$

b) How many numbers between 1 and 280 are relatively prime to 280? [4]

c) Show that any subset of $n+1$ distinct integers between 2 and $2n$ ($n \geq 2$) always contains a pair of integers with no common divisor. [6]

Q7) a) How many ways are there to distribute eight different toys among four children if the first child gets at least two toys? [6]

b) Solve the recurrence relation [6]

$$a_n = a_{n-1} + 3n^2 \text{ with } a_0 = 10$$

c) How many arrangements of the letters in Mathematics are there in which TH appear together but the TH is not immediately followed by an E? [4]

Q8) a) Seven dwarfs $D_1, D_2, D_3, D_4, D_5, D_6, D_7$, each must be assigned to one of seven jobs in a mine, $J_1, J_2, J_3, J_4, J_5, J_6, J_7$. If D_1 cannot do jobs J_1 or J_3 ; D_2 cannot do J_1 or J_5 ; D_4 cannot do J_3 or J_6 ; D_5 cannot do J_2 or J_7 ; D_7 cannot do J_4 ; D_3 and D_6 can do all jobs. How many ways are there to assign the dwarfs to different jobs? [8]

b) Find and solve a recurrence relation for the number of ways to arrange flags on an n -foot flagpole using three types of flags: red flags 2 feet high, yellow flags 1 foot high and blue flags 1 foot high. [8]



[5121]-43

M.A./M.Sc. (Semester - IV)

MATHEMATICS

MT - 803 : Differential Manifolds

(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) Attempt any five questions.
- 2) Figures to the right indicate full marks.

Q1) a) Let F be a k -tensor. With usual notation, if $AF = \sum_{\sigma \in S_k} (\text{sign} \sigma) F^\sigma$, then prove that AF is an alternating tensor. Find AF if F is already alternating. [7]

b) Show that $g(X, Y, Z) = \det \begin{pmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{pmatrix}$ is an alternating 3-tensor on \mathbb{R}^n .

Further, express g as a combination of elementary tensors. [6]

c) Define : Volume of parametrized surface in \mathbb{R}^n . [3]

Q2) a) Let U be an open set in \mathbb{R}^n and $f : U \rightarrow \mathbb{R}^n$ be of class C^1 . Let $M = \{x : f(x) = 0\}$ and $N = \{x : f(x) \geq 0\}$. If M is non-empty and $Df(x)$ has rank one at each point of M , then prove that N is an n -manifold in \mathbb{R}^n and $\partial N = M$. [8]

b) Define an exact form and give one example. [4]

c) Give an example of a 2-manifold in \mathbb{R}^3 without boundary. [4]

Q3) a) Define the differential operator d . For any k -form w , show that $d(dw) = 0$. [7]

b) If $w = x^2 dx + ydy + ze^x dz$ and $\eta = y \cos x dx + xdy + 2xy dz$, then find $(w \wedge \eta)$ [5]

c) Define 'Alternating Tensor' and give an example. [4]

- Q4)** a) Define orientation of a manifold M and induced orientation on ∂M . [4]
 b) State Green's theorem for compact, oriented 2-manifold. [4]
 c) Let $\alpha : (0,1)^2 \rightarrow \mathbb{R}^3$ be given by $\alpha(u,v) = (u,v, u^2 + v^2 + 1)$ [8]
 Let Y be the image set of α .

Evaluate $\int_Y x_2 dx_2 \wedge dx_3 + x_1 x_3 dx_1 \wedge dx_3$

- Q5)** a) Let M be a k -manifold in \mathbb{R}^n . If ∂M is non-empty, then prove that ∂M is $k-1$ manifold without boundary. [7]
 b) If $w = x^2 z^2 dx + 2(\cos y)^z dy + e^z dz$, then find dw . [5]
 c) Show that a unit n -ball B^n is an n -manifold in \mathbb{R}^n
 What is its boundary? [4]

- Q6)** a) With usual notation, show that $\alpha^*(dw) = d(\alpha^*w)$ [8]

- b) Let $A = \mathbb{R}^2 - \{0\}$. If $w = \frac{xdx + ydy}{x^2 + y^2}$, [8]

then show that w is closed and exact on A . [8]

- Q7)** a) If $T : V \rightarrow W$ is a linear transformation, and f, g are alternating tensors on W ; then prove that $T^*(f \wedge g) = T^*f \wedge T^*g$. [8]

- b) Let $w = y^2 z dx + x^2 z dy + x^2 y dz$ and $\alpha(u,v) = (u-v, uv, u^2)$ Find $\alpha^*(dw)$. [8]

- Q8)** a) State Stokes' theorem [4]

- b) Define closed form and give an example [4]

- c) What is the dimension of the space of alternating k -tensors on an n -dimensional vector space V , denoted as $A^k(V)$? Justify your answer. [8]



Total No. of Questions : 8]

SEAT No. :

P1245

[Total No. of Pages : 2

[5121]-44
M.A./M.Sc. (Semester - IV)
MATHEMATICS
MT - 804 : Algebraic Topology
(2008 Pattern)

Time : 3 Hours]

[Max. Marks : 80

Instructions to the candidates:

- 1) *Solve any five questions.*
- 2) *Figures to the right indicate marks.*

- Q1)** a) Prove that the relation of being homotopic to A is an equivalence relation. [6]
b) Let f and g be homotopic mappings of X into Y and h be a continuous mapping of Y into Z . Prove that hf and hg are homotopic mappings of X into Z . [5]
c) Let $f, g : X \rightarrow S^n$ be continuous mappings such that $f(x) \neq -g(x)$ for all $x \in X$. Show that f is homotopic to g . [5]
- Q2)** a) Prove that if Y is contractible, then every continuous mapping $f : X \rightarrow Y$ is homotopic to a constant. [6]
b) Define strong deformation retract. Show that S^n is a strong deformation retract of $\mathbb{R}^{n+1} - 0$. [5]
c) Show that a retract of a Hausdorff space is a closed subset. [5]
- Q3)** a) Prove that a non-empty open connected subset of \mathbb{R}^n is path connected. [6]
b) Suppose f, g, h are three paths such that $f * g$ and $g * h$ exist. Prove that $(f * g) * h$ and $f * (g * h)$ exist and are homotopic to each other relative to $\{0, 1\}$. [5]
c) Let $A = \{(x, y) \in \mathbb{R}^2 : x = 0, -1 \leq y \leq 1\}$ and
 $B = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1, y = \cos \pi x\}$
Show that $F = A \cup B$ is connected but not path connected. [5]
- Q4)** a) Show that $\pi_1(X, x_0)$ is a group. [6]
b) Prove that if $f : X \rightarrow Y$ is continuous, then there exists a homomorphism $f^* : \pi_1(Y, f(x_0)) \rightarrow \pi_1(X, x_0)$ where x_0 is any point in X . [5]
c) Show that the fundamental group of real projective plane is cyclic group of order two. [5]

P.T.O.

- Q5)** a) Prove that the fundamental group $\pi(S^1)$ of the circle S^1 is isomorphic to the additive group \mathbb{Z} of integers. [6]
- b) Using algebraic topology techniques, prove that every non-constant complex polynomial has a root. [5]
- c) Find the fundamental groups of \mathbb{R}^2 , $\mathbb{R}^2 - 0$ and $S^1 \times \mathbb{R}$. [5]
- Q6)** a) Prove that a covering map is a local homomorphism but the converse is not true. [6]
- b) Give an example of a non-identity covering map from S^1 to S^1 . Also give a covering map from \mathbb{R}^2 to $S^1 \times S^1$. [5]
- c) Let $p : \tilde{X} \rightarrow X$ and $q : \tilde{Y} \rightarrow Y$ be covering maps. Show that $p \times q : \tilde{X} \times \tilde{Y} \rightarrow X \times Y$ is a covering map. [5]
- Q7)** a) When does a map called a fibration? Give an example of a fibration. Prove that if $p : E \rightarrow B$ a fibration, then any path f in B with $f(0) \in p(E)$ can be lifted to a path in E . [6]
- b) Let $p : \tilde{X} \rightarrow X$ be a fibration with unique path lifting. Suppose that f and g are paths in \tilde{X} with $f(0) = g(0)$ and pf is path homotopic to pg , then f is path homotopic to g . [5]
- c) Let $p : E \rightarrow B$ a fibration. Prove that $p(E)$ is an union of path connected components of B . [5]
- Q8)** a) State and prove Brouwer fixed point theorem for B^n ($n \geq 1$). [6]
- b) Prove that every complex has a barycentric subdivision. [5]
- c) Prove that every simplex is a complex. [5]



Total No. of Questions : 8]

SEAT No. :

P1246

[Total No. of Pages : 3

[5121]-45

M.A./M.Sc. (Semester - IV)

MATHEMATICS

MT - 805 : Lattice Theory

(2008 Pattern)

Time : 3 Hour]

[Max. Marks : 80

Instructions to the candidates:

- 1) *Attempt any five questions.*
- 2) *Figures to the right indicate full marks.*

Q1) a) Let N_0 be the set of all non-negative integers. Define $m \leq n$ if and only if there exists $k \in N_0$ such that $n = km$.

Prove that N_0 is a lattice under this relation. [6]

b) Let L be a lattice then prove that I is a proper ideal of L if and only if there is a join-homomorphism ϕ of L onto C_2 such that $I = \phi^{-1}(0)$. [6]

c) Prove that every meet homomorphism is isotone. Is the converse true? Justify? [4]

Q2) a) Prove that every homomorphic image of a lattice L is isomorphic to a suitable quotient lattice of L . [6]

b) Let L be a lattice and let I be nonempty subset of L . Prove that I is an ideal if and only if $a, b \in I$ implies that $avb \in I$, and $a \in I, x \in L, x \leq a$ imply that $x \in I$. [5]

c) Let θ be a congruence relation of lattice L Then prove that for every $a \in L$. $[a]\theta$ is a convex sublattice. [5]

Q3) a) Let L and K be the lattices. Let θ be a congruence relation of L and let ϕ be a congruence relation of K . Define the relation $\theta \times \phi$ on $L \times K$ by [7]
 $\langle a, b \rangle \equiv \langle c, d \rangle (\theta \times \phi)$ if and only if $a \equiv c (\theta)$ and $b \equiv d (\phi)$.

Then prove that $\theta \times \phi$ is a congruence relation on $L \times K$. Also, show that every congruence relation of $L \times K$ is of this form.

P.T.O.

- b) Prove that the following inequalities hold in any lattice. [5]
- i) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$
- ii) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee (x \wedge z))$
- c) Prove that the dual of a distributive lattice is distributive [4]
- Q4** a) Let L be a pseudocomplemented meet semilattice, and $S(L) = \{a^*/a \in L\}$
 Then prove that [6]
- i) $a \in S(L)$ if and only if $a = a^{**}$
- ii) $a, b \in S(L)$ implies that $a \wedge b \in S(L)$.
- b) Prove that in a distributive lattice L , if the ideals IVJ and $I \wedge J$ are principal, then so are I and J . [6]
- c) Prove that every distributive lattice is modular, but not conversely. Find the smallest modular but nondistributive lattice. [4]
- Q5** a) Prove that a lattice L is modular if and only if it does not contain a pentagon. [8]
- b) Let L be a pseudocomplemented meet-semilattice and let $a, b \in L$. Then verify the formulas [4]
- $$(a \wedge b)^* = (a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$$
- c) Prove that every complete lattice is bounded. Is the converse true? Justify. [4]
- Q6** a) Let L be an finite distributive lattice. Then prove that the map $\phi : a \rightarrow r(a)$ is an isomorphism between L and $H(J(L))$, the set of all hereditary subsets of the set of all nonzero join-irreducible elements of L . [8]
- b) Prove that every modular lattice satisfies both the upper and the lower covering conditions. [5]
- c) Give an example of a lattice which is semi-modular but not modular. [3]

- Q7)** a) State and prove fixed point theorem for complete lattices. [8]
- b) Let L be a lattice of finite length. If L is semimodular, then prove that any two maximal chains of L are of the same length. [8]
- Q8)** a) Prove that a lattice is distributive if and only if it is isomorphic to a ring of sets. [8]
- b) Let L be a lattice and $a, b \in L$. If $a \wedge b$ in L and $b \vee a$ in the dual of L . Then prove that $[a \wedge b, b] \cong [a, a \vee b]$ [8]

