Total No. of Questions :8]

SEAT No. :

P1795

[Total No. of Pages :2

[5321] - 101

M.A/M.Sc.

MATHEMATICS

MT - 501 : Real Analysis

(2013 Pattern) (Semester - I) (Credit System)

Time : 3 Hours] *[Max. Marks :50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Define exterior measure of $E \subset \mathbb{R}^d$ and show that exterior measure is countably subadditive. **[5]**

b) Show that E is measurable if $m_*(E) = 0$ **[3]**

c) Define cantor set and state any two characterisitics of it. **[2]**

Q2) a) Suppose $E_1, E_2, ...$ are measurable subsets of $\mathbb{R}^d$. If $E_K \searrow E$ and M $(E_K) < \infty$ for some K then prove

that, $m(E) = \lim_{N\to\infty} m(E_N)$. What happens if $m(E_K) = \infty$? **[5]**

b) Show that there exists B closed sets A and B in $\mathbb{R}^d, d \geq 1$, with $m(A) = 0$, M (B) = 0 but $m(A + B) > 0$.

Where $A + B = \{x \in \mathbb{R}^d / x = x' + x'' \text{ with } x' \in A, x'' \in B\}$ **[3]**

c) State lusin's theorem. **[2]**

Q3) a) State and prove baunded convergence theorem. **[5]**

b) Give an example of two measurable sets such that their addition is not a measurable set. **[3]**

c) Show that lebesgue measure is translation invariant. **[2]**

P.T.O.

Q4) a) Give an example of a non- measurable set. **[5]**

b) State Little wood's three principles. **[3]**

c) Show that every continuous function is measurable. **[2]**

Q5) a) State and prove fatou's lemma. **[5]**

b) Give an example of a measurable function which is not continuous. Justify your answer. **[3]**

c) If $f(x)=\frac{1}{2}+\sin x, 0\leq x<2$ then find $f^{+}$and f^{-} **[2]**

Q6) a) Prove that the vector space L^1 is complete in it's norm. **[5]**

b) Define convolution of two functions *f* and *g*. state Poisson Kernel for a disc and fejer Kernel. **[3]**

c) Show that $X_{A\cup B}=X_A+X_B-X_{A\cap B}$ **[2]**

Q7) a) State and prove Lebesgue differentiation theorem. **[5]**

b) Define bounded variation for a function ϕ. show that if ϕ is of bounded variation then it is bounded also. Whether converse holds? Justify. **[5]**

Q8) a) State and prove rising sun lemma. **[5]**

b) Define lebesgue integral of a measurable function *f* on $\mathbb{R}^d$. show that if $f\leq g$ *a.e.* and both *f* and *g* are integrable then $\int f\leq\int g$. **[5]**

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Total No. of Questions : 8]

SEAT No. :

P1796

[Total No. of Pages : 3

[5321] - 102

M.A./M.Sc.

MATHEMATICS

MT-502: Advanced Calculus

(2013 Pattern) (Credit System) (Semester - I)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicates full marks.***

Q1) a) Define continuity of vector field. Prove that a linear transformation $\bar{f}:\mathbb{R}^n \to \mathbb{R}^m$ is continuous at each $\bar{a}\in\mathbb{R}^n$. **[4]**

b) If $\lim_{(x,y)\to(a,b)} f(x,y)=L,$ and if the one dimensional limits $\lim_{x,\to a} f(x,y)$ and $\lim_{y\to b} f(x,y)$ both exist, prove that **[4]**

$$\lim_{x\to a}\left[\lim_{y\to b} f(x,y)\right]=\lim_{y\to b}\left[\lim_{x\to a} f(x,y)\right]=L$$

c) A scalar field *f* is defined on $\mathbb{R}^n$ by the equation $f(\bar{x})=\|\bar{x}\|^2$ for all $\bar{x}$ in $\mathbb{R}^n$. Compute $f'(\bar{a};\bar{y})$ for arbitrary $\bar{a}$ and $\bar{y}$. **[2]**

Q2) a) State and prove the chain rule for derivatives of vector field in matrix form. **[5]**

b) Let $u=\dfrac{x-y}{2}$ and $v=\dfrac{x+y}{2}$ it changes f $(u,\ v)$ into f $(x,\ y)$. Use an appropriate form of the chain rule to express the partial derivatives $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$ in terms of the partial derivatives $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$. **[3]**

c) Find the directional derivative of the scalar field $f(x,y)=x^2-3xy$ along the parabola $y=x^2-x+2$ at the point (1, 2). **[2]**

P.T.O.

Q3) a) Define a line integral of a vector field. Show that a line integral remains unchanged under a change of parameter that preserves orientation. **[5]**

b) Evaluate the line integral of the vector field $\overline{f}(x,y)=(2a-y)\overline{i}+x\overline{j}$ along the path describe by $\overline{\alpha}(t)=a(t-\sin t)\overline{i}+a(1-\cos t)\overline{j}\ \ 0\le t\le 2\pi$. **[3]**

c) A force field in 3-space is given by $\overline{f}(x,y,z)=x\overline{i}+y\overline{j}+(xz-y)\overline{k}$. Compute the work done by this force in moving a particle from (0, 0, 0) to (1, 2, 4) along the line segment joining these two points. **[2]**

Q4) a) Let $\overline{f}$ be a vector field continuous on an open connected set S in $\mathbb{R}^n$. Show that if line integral of $\overline{f}$ is zero around every piecewise smooth closed path in s then the line integral of $\overline{f}$ is independent of the path in S. **[4]**

b) Let ϕ be a differentiable scalar field with a continuous gradient $\nabla\varphi$ on an open connected set S in $\mathbb{R}^n$. Prove that for any two points $\overline{a}$ and $\overline{b}$ joined by piecewise smooth path $\overline{\alpha}$ in S $\int_{\overline{a}}^{\overline{b}}\nabla\varphi\cdot d\overline{\alpha}=\varphi(\overline{b})-\varphi(\overline{a})$. Is the result independent of the path in S? Justify. **[4]**

c) Let S be the set of all $(x,y)\neq(0,0)$ in $\mathbb{R}^2$, and let $\overline{f}$ be the vector field defined on S by the equation. **[2]**

$\overline{f}(x,y)=\frac{y}{x^2+y^2}\overline{i}+\frac{x}{x^2+y^2}\overline{j}$. Show that $D_1 f_2 = D_2 f_1$ everywhere on S but $\overline{f}$ is not a gradient on S.

Q5) a) Define doubel integral of a bounded function over a rectangle. Prove that every function *f* which is bounded on a rectangle Q has a lower $\underline{I}(f)$ and an upper integral $\overline{I}(f)$, further prove that *f* is integrable over © if and only if its upper and lower integrals are equal. **[5]**

b) Use Green's theorem to compute the work done by the force field $\overline{f}(x,y)=(y+3x)\overline{i}+(2y-x)\overline{j}$ in moving a particle once around the ellipse $4x^2+y^2=4$ in the counter - clockwise direction. **[3]**

c) Let $x=\rho\cos\varphi\sin\varphi\ y=\rho\sin\theta\sin\varphi\ z=\rho\cos\varphi, \rho>0, 0\le\theta<2\pi$ and $0\le\varphi<\pi$. Show that $J(\rho,\theta,\varphi)=-\rho^2\sin\varphi$. **[2]**

Q6) a) Define a simple parametric surface. If $T=[0,2\pi]\times\left[0,\frac{\pi}{2}\right]$ under the map $\bar{r}(u,v)=a\cos u\cos v\bar{i}+a\sin u\cos v\bar{j}+a\sin v\bar{k}$ maps to a surface S, find singular points of this surface. Also explain whether S is simple. **[4]**

b) Define area of parametric surface and find the area of the hemisphere $x^2+y^2+z^2=a^2, z\geq 0$. **[4]**

c) Let S be a parametric surface whose vector representation is

$\bar{r}(u,v)=(u+v)\bar{i}+(u-v)\bar{j}+(1-2u)\bar{k}$

Find the fundamental vector product and the unit normal to the surface. **[2]**

Q7) a) State only the general formula for change of variables in double integrals. Explain the notation used. **[5]**

b) Transform the given, integral to one or more iterated integrals in polar co-ordinates **[5]**

$$\int_0^1\left[\int_0^1 f(x,y)dy\right]dx.$$

Q8) a) State and prove Gauss divergence theorem. **[5]**

b) Let S be the surface of the unit cube $0\leq x\leq 1\ \ 0\leq y\leq 1\ \ 0\leq z\leq 1$, and let $\bar{n}$ be the unit outer normal to S. If $\bar{F}(x,y,z)=x^2\bar{i}+y^2\bar{j}+z^2\bar{k}$, use the divergence theorem to evaluate the surface integral $\iint \bar{F}\cdot\bar{n}\, ds$. **[5]**

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Total No. of Questions :8]

SEAT No :

P 1797

[Total No. of Pages :2

[5321]-103

M.A/M.Sc.

MATHEMATICS

MT-503 : Group Theory

(2013 Pattern) (Semester - I) (Credit System)

Time : 3 Hours] ***[Max. Marks : 50***

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Let a and b be any elements of an abelian group and let n be any integer then show that $(ab)^u = a^u b^u$. Is this also true for non-abelian groups? **[5]**

b) Check whether set of complex roots of unity forms a group under multiplication. **[3]**

c) Prove that in a group G, there is only one identity element. **[2]**

Q2) a) State and prove fundamental theorem of cyclic groups. **[5]**

b) Find centralizer of all elements in D_4. **[3]**

c) Find a cyclic subgroup of order 4 in U(40). **[2]**

Q3) a) If $e = \beta_1, \beta_2 ---\beta_\gamma$, where e is identity permutation and $\beta_i'^s$ are 2-cycles then prove that γ is even. **[4]**

b) Determine subgroup lattice for Z_{12}. **[3]**

c) Prove that group of order P^2, where P is prime is abelian. **[3]**

Q4) a) Show that for every positive integer *n*, *Aut* (Z_n) if isomorphic to *V(n)*. **[5]**

b) Prove that D_4 cannot be written as internal direct product of two subgroups. **[3]**

c) Write all symmetries of a square. **[2]**

Q5) a) Prove that the rotations of a cube is isomorphic to S_4. **[4]**

b) Determine the number of elements of order 5 in $Z_{25} \oplus Z_5$. **[3]**

c) Prove that $G_1 \oplus G_2$ is isomorphic to $G_2 \oplus G_1$. **[3]**

Q6) a) Prove that, if |G| p^n, where p is a prime then G has non-trivial center. **[4]**

b) Let G be a group and Z(G) be the center of G. If G|Z(G) is cyclic then prove that G is abelian group. **[3]**

c) Let ϕ be a group homomorphism from G to $\overline{G}$. Prove that ker ϕ is a normal subgroup of G. **[3]**

Q7) a) Let G = {1, 8, 17, 19, 26, 28, 37, 44, 46, 53, 62, 64, 71, 73, 82, 89, 91, 98, 107, 116, 118, 127, 134} be an abelian group under multiplication modulo 135. Express G as direct product of cyclic groups. **[5]**

b) Determine group of order 255. **[5]**

Q8) a) If G is a group of order pq, where p and q are primes, p < q and p does not divide q-1 then prove that G is cyclic. **[5]**

b) Show that, Q, the group of rational numbers under addition has no proper subgroup of finite index. **[5]**

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Total No. of Questions : 8]

SEAT No :

P 1798

[Total No. of Pages :4

[5321]-104
M.A./M.Sc.
MATHEMATICS
MT - 504: Numerical Analysis
(2013 Pattern) (Semester-I) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Let f be twice continuously differentiable function on the interval $[a, b]$ with $p \in [a, b]$ and $f(p) = 0$. Further, suppose that $f'(p) \neq 0$. Show that there exists $\delta > 0$ such that for $p_0 \in \mathrm{I} = [p-\delta,\ p+\delta]$, the sequence $\{p_n\}$ generated by Newton's method converges to p. **[5]**

b) Compute each of following limits and determine the corresponding rate of convergence. **[3]**

i) $\lim_{n\to\infty} \dfrac{n-1}{n^3+2}$

ii) $\lim_{n\to\infty} \left(\sqrt{n+1}-\sqrt{n}\right)$

c) Show that $x = \sqrt{a}$ is fixed point of function $g(x) = \dfrac{x^3+3xa}{3x^2+a}$. **[2]**

Q2) a) Show that, the function e^{-x} has unique fixed point near $x = 0.6$ by fixed point. Iteration method and starting value $p_0 = 0$. (Perform at least five iterations). **[5]**

b) Verify that the equation $x^3+2x^2-3x-1=0$ has a root on the interval (1, 2), and perform five iterations by Secant method, using $p_0 = 2$ and $p_1 = 1$. **[3]**

c) Show that the order of convergence of Newton's method is two. **[2]**

P.T.O.

Q3) a) Solve the following system using Gaussian Elimination with scaled partial pivoting. **[5]**

$$3x_1 + x_2 + 4x_3 - x_4 = 7$$
$$2x_1 - 2x_2 - x_3 + 2x_4 = 1$$
$$5x_1 + 7x_2 + 14x_3 - 8x_4 = 20$$
$$x_1 + 3x_2 + 2x_3 + 4x_4 = -4$$

b) Determine the Crout decomposition of the given matrix and then solve the system Ax = b for the right hand side vector, where **[3]**

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 7 & 9 \\ 5 & 8 & -2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

c) Compute the condition number k_∞ for the matrix $A = \begin{bmatrix} 1 & -2 \\ -0.99 & 1.99 \end{bmatrix}$. **[2]**

Q4) a) Solve the following system of equations by Jacobi method. Start with $X^\circ = [0 \quad 0 \quad 0]^T$. Perform three iterations only. **[5]**

$$5x_1 + x_2 + 2x_3 = 10$$
$$-3x_1 + 9x_2 + 4x_3 = -14$$
$$x_1 + 2x_2 - 7x_3 = -33$$

b) Explain the QR algorithm for finding eigen values of symmetric tridigonal matrix. **[3]**

c) Show that, the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ has no LU decomposition. **[2]**

Q5) a) Solve the following non-linear system of equations by Newton's method with $X^{(0)} = [1 \quad 1 \quad 1]^T$. Perform two iterations only. **[5]**

$$x_1^3 - 2x_2 - 2 = 0$$
$$x_1^3 - 5x_3^2 + 7 = 0$$
$$x_2 x_3 - 1 = 0$$

b) Construct the Householder matrix H for $W = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}^T$. **[3]**

c) For the following system of non-linear equations write out the vector valued function F associated with system and compute the Jacobian of F.

$$1 + x_2 + e^{-x_1} = 0\sqrt{b^2 - 4ac}$$
$$x_1^3 - x_2 = 0$$
[2]

Q6) a) Derive the forward difference approximation for the second derivative $f''(x) \approx \dfrac{f(x_0) - 2f(x_0 + h) + f(x_0 + 2h)}{h^2}$. What is the error term associated with this formula? Numerically verify the order of approximation using $f(x) = e^x$ and $x_0 = 0$. **[5]**

b) Evaluate the integral $\int_0^1 \frac{1}{1+x^2} dx$ using Trapezoidal rule. **[2]**

c) Determine the values of coefficients A_0, A_1 and A_2 so that the quadratic formula $I(f) = \int_{-1}^{1} f(x)dx = A_0 f\left(-\frac{1}{2}\right) + A_1 f(0) + A_2 f\left(\frac{1}{2}\right)$ has degree of precision at least two. **[3]**

Q7) a) Determine degree of precision of Simpson's $\frac{3}{8}$th rule. **[5]**

b) Use Householders method to reduce the following symmetric matrix to tridigonal form. **[5]**

$$A = \begin{bmatrix} -1 & -2 & 1 & 2 \\ -2 & 3 & 0 & -2 \\ 1 & 0 & 2 & 1 \\ 2 & -2 & 1 & 4 \end{bmatrix}$$

Q8) a) Apply Euler's method to approximate the solution of initial value problem

$$\frac{dx}{dt} = 1 + \frac{x}{t},\ 1 \le t \le 6,\ x(1) = 1.$$ **[5]**

b) Define the following: **[5]**

i) Rate of Convergence.

ii) Householder matrix.

iii) The degree of precision.

iv) Triangular matrix.

v) Orthogonal matrix.

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Total No. of Questions :8]

SEAT No. :

P1799

[Total No. of Pages : 3

[5321]-105
M.A./M.Sc.
MATHEMATICS
MT-505 : Ordinary Differential Equations
(2013 Pattern) (Semester-I) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) If $y_1(x)$ is one solution of the differential equation $y'' + P(x)y' + Q(x)y = 0$, then find the other solution. **[5]**

b) Find the particular solution of $y'' + y = \text{cosec}\, x$ by variation of parameter method. **[3]**

c) Show that $y = C_1e^x + C_2e^{2x}$ is general solution of $y'' - 3y' + 2y = 0$ on any interval. **[2]**

Q2) a) Discuss the method of undetermined coefficients to find the solution of second order differential equation with constant coefficients. **[5]**

b) Verify that $y_1 = x^2$ is one solution of $x^2y'' + xy' - 4y = 0$, and find y_2 and the general solution. **[3]**

c) Replace the differential equation $y'' - x^2y' - xy = 0$ by an equivalent system of first order equations. **[2]**

Q3) a) State and prove sturm separation theorem. **[5]**

b) Let $u(x)$ be non-trivial solution of $u'' + q(x)u = 0$ where $q(x) > 0$ for all $x > 0$. If $\int_1^\infty q(x)dx = \infty$, then prove that $u(x)$ has infinitely many zeros on the positive x-axis. **[3]**

c) Classify the singular points on the x-axis of
$x^2(x-1)y'' - 2(x-1)y' + 3xy = 0$. **[2]**

P.T.O.

Q4) a) Find the general solution of $y'' + xy' + y = 0$ in terms of power series in x. **[5]**

b) Find the indicial equation and its roots of the differential equation $x^3y'' + (\cos 2x - 1)y' + 2xy = 0$. **[3]**

c) Find the power series solution of differential equation $y' = 2xy$. **[2]**

Q5) a) Find two independent Frobenius series solutions of the differential equation $xy'' + 2y' + xy = 0$. **[5]**

b) Show that $e^x = \lim_{b\to\infty} F\left(a, b, a, \frac{x}{b}\right)$. **[3]**

c) For the following system **[2]**

$$\frac{dx}{dt} = y(x^2 + 1)$$

$$\frac{dy}{dt} = -x(x^2 + 1)$$

i) Find the critical point

ii) Find the differential equation of the path.

iii) Solve the equation to find the path.

Q6) a) Solve the following system **[5]**

$$\frac{dx}{dt} = -3x + 4y$$

$$\frac{dy}{dt} = -2x + 3y$$

b) Determine the nature of the point $x = \infty$ for the equation $x^2y'' + xy' + (x^2 - 4)y = 0$. **[3]**

c) State Picards existence and uniqueness theorem. **[2]**

Q7) a) Find the general solution of differential equation

$$x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0 \text{ near the singular point } x = 0.$$ **[5]**

b) If m_1 and m_2 are roots of auxiliary equation of the system

$$\frac{dx}{dt} = a_1x + b_1y$$

$$\frac{dy}{dt} = a_2x + b_2y$$

Which are real, distinct and opposite sign, then prove that the critical point (0, 0) is a saddle point. **[5]**

Q8) a) Show that the function $f(x, y) = xy^2$ satisfies Lipschitz condition on any rectangle $a \le x \le b$ and $c \le y \le d$; but it does not satisfy a Lipschitz condition on any strip $a \le x \le b$ and $-\infty < y < \infty$. **[5]**

b) Find the exact solution of initial value problem $y' = y^2, y(0) = 1$ starting with $y_0(x) = 1$. Apply Picard's method to calculate $y_1(x), y_2(x), y_3(x)$ and compare it with the exact solution. **[5]**

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Total No. of Questions : 8]

SEAT No. :

P1800

[Total No. of Pages : 3

[5321]-201
M.A./M.Sc. (Mathematics)
MT - 601 : COMPLEX ANALYSIS
(2013 Pattern) (Semester - II) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) If $\Omega_1 \supset \Omega_2 \supset \ldots \supset \Omega_n \supset \ldots$ is a sequence of non-empty compact sets in $\mathbb{C}$ with the property that $diam(\Omega_n) \to 0$ as $n \to \infty$, then prove that there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all *n*. **[4]**

b) State and prove Schwarz reflection principle. **[4]**

c) Suppose that a function *f* is continuous in a region Ω. Prove that any two primitives of *f* (if they exist) differ by a constant. **[2]**

Q2) a) If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω, then prove that $\int_T f(z)dz = 0$ whenever *f* is holomorphic in Ω. **[5]**

b) Show that if $|a| < r < |b|$, then $\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$, where γ denotes the circle centered at the origin, of radius, *r*, with the positive orientation. **[3]**

c) State the argument principle. **[2]**

Q3) a) Consider the function f defined by $f(x+iy) = \sqrt{|x||y|}$, whenever $x, y \in \mathbb{R}$. Show that f satisfies the Cauchy-Riemann equations at the origin, but f is not holomorphic at 0. **[5]**

b) Suppose that a function $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$. of the image of *f* satisfies $2|f'(0)| \le d$. Moreover, equality holds precisely when *f* is linear, i.e. $f(z) = a_0 + a_1 z$. **[3]**

c) Show that if the real part of an entire function f is bounded, then f is constant. **[2]**

P.T.O.

Q4) a) Suppose that U and V are open sets in the complex plane. Prove that if $f: \mathrm{U} \to \mathrm{V}$ and $g: \mathrm{V} \to \mathbb{C}$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables, x and y), and $h = gof$ then $\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\frac{\partial \overline{f}}{\partial z}$. **[4]**

b) Let Ω be an open subset of $\mathbb{C}$ and let $\mathrm{T} \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then $\int_{\mathrm{T}} f(z)dz = 0$. **[3]**

c) Prove that $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$. **[3]**

Q5) a) Suppose that f is holomorphic function in a connected open set Ω, has a zero at a point $z_0 \in \Omega$, and does not vanish identically in Ω. Then prove that there exists a neighborhood $\mathrm{U} \subset \Omega$ of z_0 a non-vanishing holomorphic function g on U, and a unique positive integer n such that $f(z) = (z - z_0)^n g(z)$ for all $z \in \mathrm{U}$. **[4]**

b) Prove that $\int_0^{\infty} \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$. **[3]**

c) Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$. **[3]**

Q6) a) Suppose that f is a nowhere vanishing holomorphic function in a simply connected region Ω.

Prove that there exists a holomorphic function g on Ω such that $f(z) = e^{g(z)}$. **[5]**

b) Prove or disprove the following statement:

For $z_1, z_2 \in \mathbb{C}, \log(z_1 z_2) = \log z_1 + \log z_2$. **[3]**

c) Suppose f is continuously complex differentiable function on a region Ω, and $\mathrm{T} \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to prove that $\int_{\mathrm{T}} f(z)dz = 0$. **[2]**

Q7) a) Suppose that f is a holomorphic function in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω. Them prove that f is identically 0. **[5]**

b) i) Prove that the slit plane $\Omega = \mathbb{C} - \{(-\infty, 0]\}$ is simply connected. **[3]**

ii) Prove that the power series $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ converges at every point of the unit circle **[2]**

Q8) a) Show that it is impossible to define a total ordering on $\mathbb{C}$. **[5]**

b) i) Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic. Prove that there exists a holomorphic function f on the unit disc such that Re $(f) = u$. **[3]**

ii) Find the radius of convergence of the series $\sum_{n=1}^{\infty} n! z^n$. **[2]**

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Total No. of Questions :8]

SEAT No. :

P1801

[Total No. of Pages : 3

[5321]-202
M.A./M.Sc.
MATHEMATICS
MT - 602 : General Topology
(2013 Pattern) (Semester - II) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) If C is an infinite subset of $\mathbb{Z}_+$, then prove that C is countably infinite. **[5]**

b) Determine, for each of the following sets, whether or not it is countable. Justify your answers. **[3]**

i) The set A of all functions $f : [0, 1] \to \mathbb{Z}_+$.

ii) The set B_n of all functions $f : \{1, 2, ..., n\} \to \mathbb{Z}_+$.

iii) The set C of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$.

c) Whether the order topology on $\mathbb{Z}_+$ is the discrete topology? Justify your answer. **[2]**

Q2) a) Let X be a set. Define the basis B for a topology on X and then define the topology on X generated by B. Further, prove that if X is any set, then the collection of all one point subsets of X is a basis for the discrete topology on X. **[5]**

b) Define the topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$. Prove that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$ are not comparable. **[3]**

c) Let $\{T_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all collections. T_α. **[2]**

P.T.O.

Q3) a) Let X be an ordered set in the order topology. Let Y be subset of X that is convex in X. Then prove that the order topology on Y is the same as the topology Y inherits as a subspace of X. **[5]**

b) Let A be a subset of the topological space X and A' be the set of all limit points of A. Then prove that $\bar{A} = A \cup A'$. **[3]**

c) Show that the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps. **[2]**

Q4) a) Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in X×X. **[5]**

b) Let X and Y be topological spaces and let f: X→Y be a map. Then prove that f is continuous if and only if for each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$. **[3]**

c) Let Y be an ordered set in the order topology. Let f, g: X→Y be continuous, show that the set $\{x \in X \mid f(x) \leq g(x)\}$ is closed in X. **[2]**

Q5) a) Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod_{\alpha \in J} X_\alpha$ have the product topology. Then show that the function f is continuous if and only if each function f_α is continuous. **[5]**

b) Let f : X→Y be a map. If the function f is continuous, then prove that for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to f(x). Also prove that the converse holds if X is metrizable. **[3]**

c) Give an example of quotient map that is neither open nor closed. Justify your answer. **[2]**

Q6) a) Prove that a finite Cartesian product of connected spaces is connected. **[5]**

b) Show that the continuous image of path connected space is path connected. **[3]**

c) Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show that there exists a point x of S^1 such that $f(x) = f(-x)$. **[2]**

Q7) a) Prove that the product of two compact spaces is compact. **[5]**

b) Define compactness and limit point compactness. Whether limit point compactness implies compactness? Justify your answer. **[5]**

Q8) a) Show that a subspace of regular space is regular. **[5]**

b) Let X be a set and D be a collection of subsets of X that is maximal with respect to the finite intersection property. Then prove that any finite intersection of elements of D is an element of D. **[5]**

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Total No. of Questions : 8]

SEAT No. :

P1802

[Total No. of Pages : 2

[5321]-203
M.A./M.Sc.
MATHEMATICS
MT - 603 : Ring Theory
(2013 Pattern) (Semester - II) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Let R be a ring with 1 such that the non-units in R form a subgroup of (R, t) Then prove that char (R) is either O or else a power of prime. **[4]**

b) Let R be a commutative ring whose characteristic is a prime p. Then show that $(a + b)^p = a^p + b^p$ for all $a, b \in$ R. **[3]**

c) Give examples of two zero-divisors in matrix ring $M_2(\mathbb{Z})$ whose sum is not a zero divisor in $M_2(\mathbb{Z})$. Justify. **[3]**

Q2) a) Let R be a non zero ring in which multiplication is non-trivial. Prove that R is division ring if and only if R has non left ideals other than (0) and R. **[5]**

b) Show that $1 + 2x + 4x^2$ is a unit in the ring $\mathbb{Z}_8[x]$. **[2]**

c) Give an example of degree one polynomial over $\mathbb{Z}_6[x]$ which has three distinct roots in $\mathbb{Z}_6$. Justify. **[3]**

Q3) a) For $n \geq 2$, prove that the ring $\mathbb{Z}/n\mathbb{Z}$ has no non-trivial nilpotent element if and only if n is squre free. **[5]**

b) Let P be a prime ideal in a commutative ring R. Prove that for any pair of ideals I and J with $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. **[2]**

c) Let p be a prime in $\mathbb{Z}$. Show that $M = \{(px, y) | x, y \in \mathbb{Z}\}$ is a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$. **[3]**

P.T.O.

Q4) a) For an ideal I in a commutative ring R, Let $\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$

Show that $\sqrt{I}$ is an ideal of R containing I. Also show that if $I \subseteq J$ then $\sqrt{I} \subseteq \sqrt{J}$. **[4]**

b) let f : R $\to$ S be a surjective ring homomorphism. Prove that inverse image of a maximal ideal in S is a maximal ideal in R. **[4]**

c) Let f : R $\to$ S, g : S $\to$ T be homomorphisms of rings. If gof is an epimorphism, then prove that 2 is an epimorphism. **[2]**

Q5) a) If the ring End_k (V) is a simple ring, the prove that V is finite dimensional vector space over the field K. **[4]**

b) Show that every Euclidean domain is a principal ideal domain. **[4]**

c) Find units of $\mathbb{Z}\left[i\sqrt{5}\right]$. **[2]**

Q6) a) Let R be a factorization domain in which every irreducible element is prime. Prove that R is UFD. **[5]**

b) With usual notation show that $\frac{Q[x]}{\langle 1+x^2 \rangle} \approx Q[i]$. **[5]**

Q7) a) For a commutative integral domain R with unity, prove that the following are equivalent. **[5]**

i) R is field.

ii) R [x] is Euclidean domain.

iii) R [x] is PID.

b) Show that the polynomial $1 + x^4$ is irreducible in $\mathbb{R}[x]$. **[3]**

c) Let S be a subring of a ring R. Show by example that S and R may both have unities but they may not be the same. **[2]**

Q8) a) Let F be a free module over R and M be a submodule of F. Prove that M is also free module and $\dim_R M \leq \dim_R F$. **[5]**

b) Show that a finitely generated module M over PID is a direct sum of its torsion part M_t and a free submodule M_f. **[5]**

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Total No. of Questions : 8]

SEAT No. :

P1803

[Total No. of Pages : 3

[5321]-204
M.A./M.Sc.
MATHEMATICS
MT - 604 : Linear Algebra
(2013 Pattern) (Semester - II) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Solve any five questions out of eight questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Let V be the Vector space of all mappings from R to R and V_1, V_2 be the subsets of V of even and odd functions respectively, that is, $V_1 = \{f \in V / f(-x) = f(x)\}$ and $V_2 = \{f \in V / f(-x) = f - (x)\}$ then show that V is a direct sum of the subspaces V_1 and V_2. **[5]**

b) Show that the only nontrivial proper subspaces of the vector space $\mathbb{R}^2$ are the "lines through the origin", that is, subsets L of the type $L = \{(x, y) \in \mathbb{R}^2 | ax + by = 0\}$ where a, b are fixed elements of $\mathbb{R}$, not both zero. **[3]**

c) Complete the set {(2, 1, 4, 3), (2, 1, 2, 0)} to form a basis of $\mathbb{R}^4$. **[2]**

Q2) a) If U, V are vector spaces over F and let $(e_1, e_2, ..., e_n)$ be an ordered basis of U. Given a list $f_1, f_2, ..., f_n$ of elements of V, then prove that there is a unique linear mapping $f : U \to V$, such that $f(e_i) = f_i, i = 1, 2, ..., n$ Further, f is an isomorphism if and only if $(f_1, f_2, ..., f_n)$ is a basis of V. **[5]**

b) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a mapping defined by

$f(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3\ x_3 - x_1)$ **[3]**

i) Show that *f* is linear mapping.

ii) What are the conditions in order that $(a, b, c) \in \operatorname{Im} f$?

iii) What are the conditions in order that $(a, b, c) \in \ker f$?

c) Find a basis of the vector space $\mathbb{C}$ over $\mathbb{R}$. **[2]**

P.T.O.

Q3) a) Let $\phi:\mathbb{C}^2 \to \mathbb{C}^2$ be a linear mapping such that $\phi(a, b) = (\alpha a + \beta b, \gamma a + \delta b)$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Prove that ϕ is invertible if and only if $\alpha\delta - \beta\gamma \neq 0$. **[5]**

b) Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping defined by $f(a,b,c) = (a, a+b, 0)$. Find the matrices A and B respectively of the linear mapping f with respect to the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where $e_1' = (1,1,0)$, $e_2' = (0,1,1)$, $e_3' = (1,1,1)$. **[3]**

c) What is the dimension of the vector space V = {P_n-polynomial of degree $\leq n$, with complex coefficients} **[2]**

Q4) a) Let U, V be finite dimensional vector spaces over F and Let $\phi: V \to U$ be a linear mapping. Then prove that, $rank(\phi) + nullity(\phi) = \dim(V)$.**[5]**

b) Let $\phi:\mathbb{C}^3 \to \mathbb{C}^3$ be a linear mapping defined by $\phi(v) = Av, v \in \mathbb{C}^3$ and the matrix of ϕ with respect to the standard basis $\mathbb{C}^3$ is A, then show that matrices $A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are similar matrices over $\mathbb{C}$. **[3]**

c) Let $V = \mathbb{R}^2$ and W be the subspace of V generated by (1,1). Write down a typical element of V/W. **[2]**

Q5) a) Let $A \in F^{n \times n}$. Suppose A has n distinct eigenvalues $\lambda_1, ..., \lambda_n$ then prove that there exists an invertible matrix P such that $P^{-1}AP = diag(\lambda_1, ..., \lambda_n)$.**[5]**

b) The three eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ of a 3 × 3 matrix A are associated respectively with eigenvalues 1, –1 and 0. Find A. **[3]**

c) Define: **[2]**

i) Orthogonal basis of vector space.

ii) Jordan Block.

Q6) a) Reduce the following matrix into triangular form $A = \begin{bmatrix} 2 & -1 & 4 \\ 2 & 1/2 & 7 \\ 4 & 2 & 8 \end{bmatrix}$. **[5]**

b) If B is symmetric bilinear form on a vector space V over a field F and let $char(\text{F}) \neq 2$ then prove that there exist an orthogonal basis of V relative to B. **[3]**

c) If the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 4 \\ -1 & 4 & 3 \end{bmatrix}$ then find quadratic form of the matrix A.

[2]

Q7) a) Given $A = \begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ find an invertible matrix P such that $P^{-1}AP$ is in Jordan Canonical form. **[5]**

b) Prove that every finite dimensional Euclidean vector space has an orthonormal basis. **[5]**

Q8) a) Prove that a quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$, $x = (x_1, x_2, ..., x_n)^t \in \mathbb{R}^n$ on R^n can be reduced to diagonal form. $Q(x) = \lambda_1 x_1'^2 + \lambda_2 x_2'^2 + ... + \lambda_n x_n'^2$. by an orthogonal transformation of coordinates, where $\lambda_1, \lambda_2 ..., \lambda_n$ are the eigenvalues of A. **[5]**

b) State and prove Sylvester's Theorem. **[5]**

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Total No. of Questions : 8]

SEAT No. :

P1804

[Total No. of Pages : 3

[5321] - 205

M.A./M.Sc.

MATHEMATICS

MT-605: Partial Differential Equations

(2013 Pattern) (Credit System) (Semester - II)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Eliminate the parameters a and b from the equaiton. **[4]**

$$z^2(1+a^3) = 8(x+ay+b)^3 .$$

b) Find the general solution of : $x(y-z)p + y(z-x)q = z(x-y)$. **[4]**

c) Define the following terms and example of each **[2]**

i) Linear equation

ii) Quasi-linear equation

Q2) a) Find the general solution of: **[4]**

$$(y^2+yz+z^2)dx + (z^2+xz+x^2)dy + (y^2+xy+x^2)dz = 0 .$$

b) Show that the equations: $xp - yq - x = 0$ and $(x^2p + q - xz) = 0$ are compatible. Also find their common solution. **[4]**

c) Find the complete integral of : $p^2 + q^2 = x + y$. **[2]**

P.T.O.

Q3) a) If $h_1 = 0$ and $h_2 = 0$ are compatible with $f = 0$, then prove that h_1 and h_2 satisfy **[4]**

$$\frac{\partial(f,h)}{\partial(x,u_x)} + \frac{\partial(f,h)}{\partial(y,u_y)} + \frac{\partial(f,h)}{\partial(z,u_z)} = 0$$

b) Find the complete integral of $px + qy = pq$ by Charpit's method. **[4]**

c) Verify that the equation is integrable: **[2]**

$yz(y+z)dx + xz(x+z)dy + xy(x+y)dz = 0$

Q4) a) Find the general integral surface of the D.E. **[4]**

$(x-y)y^2 p + (y-x)x^2 q = (x^2 + y^2)z$ and the particular solution through curve C: $x\,z = a^2, y = 0$.

b) Find the method of characteristic, the integral Surface of $p\,q = z$ which passes through curve: $x_0 = 0, y_0 = s, z_0 = s^2$. **[4]**

c) Derive the analytic expression for the Monge cone at (x_0, y_0, z_0). **[2]**

Q5) a) Verify that the equation is integrable and find the corresponding integral: **[4]**

$$(y^2 + yz)dx + (z^2 + xz)dy + (y^2 - xy)dz = 0.$$

b) Find d'Alembert's solution of one dimensional wave equation which describes the vibration of infinite string. **[3]**

c) Reduce the equation: **[3]**

$U_{xx} + 2U_{xy} + 17U_{yy=0}$ to canonical form and solve it.

Q6) a) State and prove Kelvin's inversion theorem. **[5]**

b) Find the solution of the Heat - equation in an infinite rod which is defined as **[5]**

$$U_t = kU_{xx}, \; -\infty < x < \infty, t > 0$$

$$U(x,0) = f(x), \; -\infty < x < \infty$$

Q7) a) If U(x, y) is harmonic in a bounded domain D and continuous in $\bar{D} = D \bigcup B$ then U attains it's maximum on the boundary B of D. **[4]**

b) State and prove Harnack's theorem. **[4]**

c) Classify the following equation into hyperbolic, parabolic or elliptic type **[2]**

$$u_{xx} + 2(1+ \propto y)u_{yz} = 0 .$$

Q8) a) Use Duhamel's principle and solve the non homogeneous wave **[5]**

equation $u_{tt} - c^2 u_{xx} = F(x,t), -\infty < x < \infty, t > 0$

with conditions: $u(x,0) = u_{t(x,0)} = 0 - \infty < x < \infty$.

b) State Dirichlet's problem fo rectangle and fine it's solutions. **[5]**

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Total No. of Questions :8]

SEAT No. :

P1805

[Total No. of Pages :3

[5321] - 301

M.A/M.Sc.

MATHEMATICS

MT - 701 : Combinatorics

(2013 Pattern) (Credit System) (Semester - III)

Time : 3 Hours] *[Max. Marks :50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) What is the probability of randomly choosing a permutation of the 10 digits 0, 1, 2, ..., 9 in which. **[5]**

i) An odd digit is in the first position and 1, 2, 3, 4 or 5 is in the last position?

ii) 5 is not in the first position and 9 is not in the last position?

b) Prove by combinatorial argument **[3]**

$$\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+....+\binom{n}{r}=\binom{n+1}{r+1}$$

Hence, evaluate the sum

$1\times 2 \times 3 + 2 \times 3 \times 4 + ...,+ (n-2)(n-1)n.$

c) Find a generating function for the number of integer solutions of $2x + 3y + 7z = r$ with $x, y, z \geq 0$ **[2]**

Q2) a) What fraction of all arrangements of 'INSTRUCTOR' have three consecutive vowels? **[5]**

b) How many ways are there to deal a 6- card hand that contsins at least one Jack, at least one 8 and at least one 2? **[3]**

c) Find two different chessboards (not row or column arrangements of one another) that have the same rook polynomial. Also, write that rook polynomial. **[2]**

P.T.O.

Q3) a) Use generating functions to find the number of ways to split 6 copies of one book, 7 copies of a second book and 11 copies of a third book between two teachers if each teacher gets 12 books and each teacher gets at least 2 copies of each book. **[5]**

b) Find ordinary generating function whose coefficient $a_r = (r+1)\,r\,(r-1)$.

Hence evaluate the sum $3 \times 2 \times 1 + 4 \times 3 \times 2 + \ldots + (n+1)\,n\,(n-1)$. **[3]**

c) How many distributions of 24 different objects into three different boxes are there with twice as many objects in one box as in the other two combined? **[2]**

Q4) a) How many ways are there to select 300 chocolate candies from seven types if each type comes in boxes of 20 and if at least one but not more than five boxes of each type are chosen? **[5]**

b) How many ways are there to distribute eight different toys among four children if the first child gets at least two toys? **[3]**

c) Solve the recurrence relation

$a_n^2 = 2\,a_{n-1}^2 + 1$ with $a_0 = 1$ **[2]**

Q5) a) How manyr - digit quaternary sequences are there in which the total number of 0's and 1's is even? **[5]**

b) How many n - digit decimal sequences (using digits 0, 1, 2, ..., 9) are there in which digits 1, 2, 3 all appear? **[3]**

c) Solve the recurrence relation

$a_n = 2a_{n/2} + 2, n \geq 4,$ with $a_2 = 1$

(assuming n is power of 2). **[2]**

Q6) a) Using generating function, solve the recurrence relation

$a_n = 2\, a_{n-1} + 2^n$, with $a_0 = 1$. **[5]**

b) How many ways are there to assign 20 different people to three different rooms with atleast one person in each room? **[3]**

c) Solve the recurrence relation

$a_n = 3\, a_{n-1} + n^2 - 3$, with $a_0 = 1$. **[2]**

Q7) a) Find recurrence relation for the number of *n* - digit binary sequences with an even number of zeros. **[5]**

b) i) Using Inclusion exclusion principle, find the number of ways to distribute 25 identical balls into 6 distinct boxes with at most 6 balls in any of the first three boxes. **[3]**

ii) How many arrangements of the letters in MATHEMATICS are there in which TH appear together but the TH is not immediately followed by an E? **[2]**

Q8) a) How many ways are there to send six birthday cards, denoted C_1, C_2, C_3, C_4, C_5, C_6, to three aunts and three uncles, denoted A_1, A_2, A_3, U_1, U_2, U_3; if aunt A_1 would not like cards C_2 and C_4; if A_2 would not like C_1 or C_5; if A_3 likes all cards, if U_1 would not like C_1 or C_5; if U_2 would not like C_4; and if U_3 would not like C_6? **[5]**

b) Find a recurrence relation for the ways to distributes n identical balls into K distinct boxes with between two and four balls in each box. Repeat the problem with balls of three colours. **[5]**

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Total No. of Questions : 8]

SEAT No :

P 1806

[Total No. of Pages : 3

[5321]-302

M.A./M.Sc.

MATHEMATICS

MT - 702 : Field Theory

(2013 Pattern) (Credit System) (Semester - III)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let F be a field and $p(x) \in F[x]$ be an irreducible polynomial. If K is an extension field of F containing a root α of p(x) and F(α) denote the subfield of K generated by α over F then prove that F(α) is isomorphic to $F[x]/\langle p(x)\rangle$. **[5]**

b) If K be any field and $\overline{K}$ denote the algebraic closure of K then prove that $k = \overline{K}$ if and only if K is algebraically closed. **[3]**

c) Determine the degree of $\theta = 2 + \sqrt{3}$ over $\mathbb{Q}$. **[2]**

Q2) a) Show that the doubling the cube and squaring the circle are impossible by using compass and straightedge. **[5]**

b) Show that $x^{p^n} - x$ is precisely the product of irreducible polynomials in $\mathbb{F}_P[x]$ of degree d, where 'd' dividing 'n'. **[3]**

c) Find the discriminant of the polynomial

$f(x) = x^3 - 2x + 4$. **[2]**

P.T.O.

Q3) a) Show that $\mathbb{Q}(2^{1/4}, i)$ is the spliting field of the polynomial $f(x) = x^4 - 2 \in \mathbb{Q}[x]$ over $\mathbb{Q}$. Also find it's degree of extension. **[5]**

b) State fundamental theorem of Galois theory. **[3]**

c) Is $\mathbb{Q}(\sqrt[4]{2})$ Galois over $\mathbb{Q}$? Justify. **[2]**

Q4) a) Prove that the spliting field of a separable polynomial is Galois. **[5]**

b) If ch(F) $\neq$ 2 then prove that the permutation $\sigma \in S_n$ is an element of A_n if and only if it fixes the square root of the discriminant D. **[3]**

c) Show that x^2-2 is separable over $\mathbb{Q}$. **[2]**

Q5) a) Show that Galois group of $x^{p^n} - x$ over $\mathbb{F}_p$ is a cyclic group of order n. **[5]**

b) Show that a polynomial $f(x) \in F[x]$ is separable if and only if $f(x)$ and $D_x(f(x))$ are relatively prime. **[3]**

c) Show that any automorphism of a field k fixes it's prime subfield. **[2]**

Q6) a) If F be any field and $f(x) \in F[x]$ then prove that there exists an extension K of F which is a splitting field for $f(x)$. **[5]**

b) Show that the polynomial $x^4 + 1$ is reducible over $\mathbb{F}_P$ for every prime P. **[3]**

c) Define the following terms. **[2]**

i) Cyclic Extension.

ii) Quadratic extension.

Q7) a) If F be a field of characteristic p then prove that $(a+b)^p = a^p + b^p$ and $(ab)^p = a^p b^p$ for all a,b, $\in$ F, and p is a prime number. **[5]**

b) Show that the n^{th} cyclotomic polynomial $\Phi_n(x)$ is irreducible over $\mathbb{Z}$. **[5]**

Q8) a) If field K generated by a finite number of algebraic elements of degree $n_1, n_2, ..., n_r$ over a field F then prove that K is algebraic field of degree $\leq n_1 n_2 .... n_r$. **[5]**

b) Show that $p(x) = x^3 - 2x - 2$ is irreducible polynomial over $\mathbb{Q}$. Also compute $(1 + \theta)(1 + \theta + \theta^2)$ in $\mathbb{Q}(\theta)$, where θ is a root of $p(x)$. **[5]**

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Total No. of Questions :8]

SEAT No :

P 1807

[Total No. of Pages :2

[5321]-303

M.A/M.Sc.

MATHEMATICS

MT-703 : Functional Analysis

(2013 Pattern) (Credit System) (Semester - III)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Prove that the Bergman space $L_a^2(G)$ is a Hilbert space. **[5]**

b) Let $M = \{\{x_n\} \in l^2 \mid$ for all but finitely many $n\}$. Is M a closed subspace of l^2? Justify your answer. **[3]**

c) State Hahn-Banach Theorem. **[2]**

Q2) a) State and prove Pythagorean Theorem. **[4]**

b) Show that the Fourier transform is a liner isometry from $L_c^2[0,2\pi]$ onto $l^2(Z)$. **[4]**

c) Define reflexive space and give an example. **[2]**

Q3) a) If A is a normal operator and λ and μ are distinct eigenvalues of A, then prove that $ker(A-\lambda) \perp ker(A - \mu)$. **[4]**

b) Let $\mathcal{H}$ be a Hilbert space and suppose f and g are linearly independent vectors in $\mathcal{H}$ with $||f|| = ||g|| = 1$. show that $||tf+(1-t)g||<1$ for $0 < t<1$. What does this say about $\{h \in \mathcal{H} \mid ||h|| \leq 1\}$? **[3]**

c) Give an example of linear map between Hilbert spaces, which is an isometry but not surjective. Justify your answer. **[3]**

P.T.O.

Q4) a) If $A \in B(\mathcal{H})$, *then prove that* $||A|| = ||A^*|| = ||A^*A||^{1/2}$. **[4]**

b) Show that for a separable Hilbert space $\mathcal{H}$, with a orthonormal basis $\{e_n\}$, if A is an operator defined by $Ae_n = \frac{1}{n}e_n$ then A is a compact operator. **[3]**

c) Let P and Q be projections, show that $P+Q$ is a projection if and only if $range(P) \perp range(Q)$ **[3]**

Q5) a) State and prove the Principle of Uniform Boundedness. **[5]**

b) If $A \in B(\mathcal{H})$, then prove that $ker\ A = (ranA^*)^{\perp}$. **[3]**

c) Give an example of a convex set in a Hilbert space. **[2]**

Q6) a) Let T be a compact normal operator. Prove that T is positive if and only if all its eigenvalues are non-negative real numbers. **[5]**

b) Show that l^{∞} is not separable. **[3]**

c) If $\mathcal{X}$ and $\mathcal{Y}$ are banach spaces and $A: \mathcal{X} \to \mathcal{Y}$ is a bounded linear transformation that is bijective, then prove that A^{-1} is bounded. **[2]**

Q7) a) State and prove the Open Mapping Theorem. **[5]**

b) i) Are c and c_0 isometrically isomorphic ? Justify your answer. **[3]**

ii) Give an example of a Banach space which is not a Hilbert space. **[2]**

Q8) a) Let X be a normed space and f b a linear functional on X. If ker f is closed then prove that f is continuous. **[5]**

b) i) If $\{x_n\}$ is a sequence in Banach space $\mathcal{X}$ such that $\sum_{n=1}^{\infty} \| x_n \| < \infty$, then prove that the series $\sum_{n=1}^{\infty} x_n$ converges in $\mathcal{X}$. **[3]**

ii) State Riesz Representation Theorem for a Hilbert space. **[2]**

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Total No. of Questions : 8]

SEAT No. :

P1808

[Total No. of Pages : 3

[5321]-401
M.A./M.Sc.
MATHEMATICS
MT - 801 : Number Theory
(2013 Pattern) (Semester - IV) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full makrs.*

Q1) a) Let a, b and $m > 0$ be given integers and $g = (a, m)$ then prove that the congruence $ax \equiv b(\text{mod}\, m)$ has a solution if and only if $g|b$. **[5]**

b) How many solutions are there to each of the following congruences: **[3]**

i) $15x \equiv 25(\text{mod } 35)$

ii) $15x \equiv 24(\text{mod } 35)$

iii) $15x \equiv 0(\text{mod } 35)$

c) Is 3 a quadratic residue of 7? Justify. **[2]**

Q2) a) Let p be an odd prime then prove the following: **[5]**

i) $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right)=\left(\frac{ab}{p}\right)$

ii) $\left(\frac{a^2}{p}\right)=1 \text{ if } (a, p)=1$

b) Prove that $n^{13}-n$ is divisible by 2, 5 and 7 for any integer n. **[3]**

c) Show that 2, 4, 6, ..., $2m$ is a complete residue system modulo m if m is odd. **[2]**

Q3) a) Let $\mu(n)$ be the Möbius *mu* function then prove that $\mu(n)$ is multiplicative and $\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1 \end{cases}$ **[5]**

b) What is the highest power of 6 dividing 533!? **[3]**

c) Find the minimal polynomial of $1+\sqrt{2}+\sqrt{3}$. **[2]**

Q4) a) Let p denotes a prime then prove that $x^2 \equiv -1 (\text{mod } p)$ has solutions if and only if $p = 2$ or $p \equiv 1 (\text{mod} 4)$. **[5]**

b) Prove that $\frac{(2n)!}{(n!)^2}$ is even if n is *a* positive integer. **[3]**

c) For any real number x prove that $[x] + \left[x + \frac{1}{2}\right] = [2x]$. **[2]**

Q5) a) Let *a, b* and *c* be integers with not both *a* and *b* equal to 0 and let $g = g.c.d\,(a, b)$. If $g \nmid c$ then show that the equation $ax + by = c$ has no solution in integers. If $g|c$ then show that $ax + by = c$ has infinitely many solutions and are of the form $x = x_1 + \frac{kb}{g}$, $y = y_1 - \frac{ka}{g}$ where k is an integer and (x_1, y_1) is one integral solution. **[5]**

b) Find the least positive integer x such that $x \equiv 5 \ (\text{mod} 7)$, $x \equiv 7 \ (\text{mod} 11)$ and $x \equiv 3 \ (\text{mod} 13)$. **[5]**

Q6) a) Prove that an algebraic number ξ satisfies a unique irreducible monic polynomial equation $g\,(x) = 0$ over Q and every polynomial equation over Q satisfied by ξ is divisible by $g\,(x)$. **[5]**

b) Prove that all solutions of $3x + 5y = 1$ can be written in the form $x = 2 + 5\,t$, $y = -1 - 3t$, where t is an integer. **[5]**

Q7) a) If the norm of α is denoted by $N(\alpha)$ then prove the following: **[5]**

i) $N(\alpha\beta) = N(\alpha)\,N(\beta)$

ii) The norm of an integer in $Q\left(\sqrt{m}\right)$ is a rational integer.

iii) If γ is an integer in $Q\left(\sqrt{m}\right)$ then $N(\gamma) = \pm 1$ if and only if γ is a unit.

b) Prove that if p is an odd prime then $x^2 \equiv 2 \pmod{p}$ has solutions if and only if $p \equiv 1$ or $7 \pmod{8}$. **[3]**

c) Find the smallest positive integer n so that $\sigma(x) = n$ has no solutions; exactly one solution; exactly two solution; exactly three solutions. **[2]**

Q8) a) For every positive integer n prove that **[5]**

i) $\sigma(n) = \prod_{p^{\alpha}\|n} \left(\frac{p^{\alpha+1} - 1}{p - 1} \right)$

ii) $\sum_{d|n} \phi(d) = n$

b) Evaluate $\phi(m)$ for $m = 10$, 11 and 12. **[3]**

c) Is $x^2 \equiv 5 \pmod{227}$ solvable? Justify. **[2]**

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Total No. of Questions : 8]

SEAT No. :

P1809

[Total No. of Pages : 3

[5321] - 402
M.A./M.Sc.
MATHEMATICS
MT - 802 : Differential Geometry
(2013 Pattern) (Semester - IV) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

SECTION-I

Q1) a) Let s be an n-surface in $\mathbb{R}^{n+1}$ and $\alpha : I \rightarrow S$ be a parameterized curve in s, Let $t_o \in I$ and $v \in S_{\alpha(t_o)}$. Then prove that there exists a unique vector field v tangent to S along α, which is parallel and has $V(t_o) = v$. **[5]**

b) Determine whether the vector field $X(x_1, x_2) = (x_1, x_2, 1+x^2, 0)$ is complete or not. **[3]**

c) Define the terms : **[2]**

i) level sets

ii) graph of function

Q2) a) Let C be a connected oriented plane curve and $\beta : I \rightarrow C$ be unit speed parameterization of C. Prove that β is either one to one or periodic. Also prove that β is periodic if and only if C is compact. **[5]**

b) Show that the Weingarten map is self-adjoint operator. **[3]**

c) State Inverse function Theorem. **[2]**

Q3) a) Find the integral curve through p = (a, b) of the vector field X(p) = (p, X(p)) where $X(x_1, x_2) = (-x_2, x_1)$. **[4]**

b) Find the global parameterization of the curve $(x_1 - a)^2 + (x_2 - b)^2 = \gamma^2$. **[3]**

c) Let, $g : I \rightarrow \mathbb{R}$ be smooth function and C denote the graph of g. Then show that the curvature of C at point (t, g(t)) is $g''(t)/\left(1+(g'(t))^2\right)^{3/2}$ for an appropriate choice of an orientation. **[3]**

P.T.O.

Q4) a) Explain why an integral curve cannot cross itself as it does for parameterized curve. **[4]**

b) Find and sketch the gradient field of the function $f(x_1, x_2) = x_1 + x_2$. **[4]**

c) Define the terms : **[2]**

i) Gauss map

ii) Weingartin map

Q5) a) Consider the vector field $X(x_1, x_2) = (x_1, x_2, 1, 0)$ on $\mathbb{R}^2$ for $t \in \mathbb{R}$ and $P \in \mathbb{R}^2$ and let $\phi_t(p) = \alpha_p(t)$ where α_p is the maximal integral curve of X through 'P' then.

i) Show that for each t, ϕ_t is 1-1 transformation from $\mathbb{R}^2$ onto itself. Geometrically, what does this transformation mean!?

ii) Show that $\phi_0 =$ identity, $\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$ $\forall t_1, t_2 \in \text{I}$ and $\phi_{-t} = \phi_t^{-1}, \forall t \in \text{R}$ **[5]**

b) Let a,b, c $\in \mathbb{R}$ be such that $ac - b^2 > 0$ then show that the maximum and minimum values of the function $g(x_1, x_2) = x_1^2 + x_2^2$ on the ellipse $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$ are of the form $1/\lambda_1, 1/\lambda_2$ Where λ_1, λ_2, are eigen values of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. **[5]**

Q6) a) State and prove Lagrange's multiplier theorem for n-surface. **[5]**

b) Let S be an n-surface in $\mathbb{R}^{n+1}$ oriented by the unit normal vector field N. Let $p \in s$ and $v \in S_p$. Then show that for every parameterized curve $\alpha : \text{I} \to \text{S}$ with $\alpha(t_o) = v$ for some $t_o \in 1$, $\ddot{\alpha}(t_o) \cdot N_{(p)} = L_p(v) \cdot v$. **[5]**

Q7) a) Show that the Möbius band is an unorientable 2-surface. **[3]**

b) Show that in parallel transport $P_\alpha : S_p \to S_q$ where S is an n-surface in $\mathbb{R}^{n+1}$, $p, q \in s$ and α is piecewise smooth parameterized curve from p to q is an one to one, onto linear map preserving the dot product. **[5]**

c) Let X be smooth vector field along the parameterized curve $\alpha : \text{I} \to \mathbb{R}^{n+1}$ and f be smooth function along $\alpha(t)$. Then prove that $(f\dot{X}) = f'X + f\dot{X}$. **[2]**

Q8) a) Let $\phi : U \to \mathbb{R}^{n+1}$ be a parameterized n-surface in $\mathbb{R}^{n+1}$ and $p \in U$ then show that there exist an open set $U_1 \subset U$ about p such that $\phi(U_1)$ is an n-surface. **[4]**

b) Let S be an n-surface in $\mathbb{R}^{n+1}$ and $f : S \to \mathbb{R}^k$. Then show that f is smooth if and only if $f_o\phi : U \to \mathbb{R}^k$ is smooth for each local parameterization $\phi : U \to S$. **[4]**

c) Define the term n-surface in $\mathbb{R}^{n+1}$ with an example. **[2]**

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Total No. of Questions : 8]

SEAT No. :

P1810

[Total No. of Pages : 4

[5321]-403

M.A/M.Sc.

MATHEMATICS

MT - 803 : Fourier Analysis and Boundary Value Problems

(2013 Pattern) (Semester - IV) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*

2) *Figures to the right indicate full marks.*

Q1) a) If $f \in C_p(0,\pi)$, then prove that the Fourier cosine series coefficient a_n tends to zero as n tends to infinity. **[5]**

b) Find the Fourier sine series for the function $f(x) = x(\pi - x)$ $(0 < x < \pi)$. **[3]**

c) Find the fourier cosine series for the function $f(x) = \sin x$ $(0 < x < \pi)$. **[2]**

Q2) a) Let f denote a function such that **[5]**

i) f is continuous on the interval $-\pi \le x \le \pi$

ii) $f(-\pi) = f(\pi)$

iii) Its derivative f' is piecewise continuous on the interval $-\pi < x < \pi$.

Prove that the Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n x + b_n \sin n x)$ for f with coefficients $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n x\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n x\,dx$ converges absoluely and uniformly to $f'(x)$ on the interval $-\pi \le x \le \pi$.

P.T.O.

b) Find the Fourier series on the interval $-\pi < x < \pi$ that corresponding to the function **[3]**

$$f(x)=\begin{cases}\dfrac{2}{\pi}x+2 & -\pi < x \le 0\\ 2 & 0 \le x \le \pi\end{cases}$$

c) If m and n are positive integers, then show that **[2]**

$$\int_0^{\pi} \cos mx \cos n\, x\, dx = \begin{cases} 0 & \text{when} \quad m \ne n \\ \dfrac{\pi}{2} & \text{when} \quad m = n\end{cases}$$

Q3) a) If f is piecewise continuous on the interval $-\pi < x < \pi$ then prove that

$$\int_{-\pi}^{x} f(s)ds = \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty}\frac{1}{n}\left\{a_n \sin n\, x - b_n\left[\text{cosnx} + (-1)^{n+1}\right]\right\}$$ **[5]**

b) Solve the following boundary value problem **[5]**

$$u_t(x,t) = k\, u_{xx}(x,t) \;\; (0 < x < c, t > 0)$$

$$u_x(0,t) = 0,\, u_x\;(c,t) = 0\;(t > 0)$$

$$u(x,0) = f(x)\;\;(0 < x < c)$$

Q4) a) Solve the following boundary value problem $\rho^2 u_{\rho\rho}(\rho,\phi) + \rho u_\rho(\rho,\phi)$ $+u_{\phi\phi}(\rho,\phi) = 0\,(1 < \rho < b, 0 < \phi < \pi)$

$$u(\rho,0) = 0,\; u(\rho,\pi) = 0\;\;(1 < \rho < b)$$

$$u(1,\phi) = 0\;\; u(b,\phi) = u_0\;\;(0 < \phi < \pi)$$

where u_0 is constant. **[5]**

b) Solve the following boundary value problem **[5]**

$$u_{xx}(x,y) + u_{yy}(x,y) = 0\;\;(0 < x < a,\, 0 < y < b)$$

$$u(0,y) = 0,\, u(a,y) = 0\;\;(0 < y < b)$$

$$u(x,0) = f(x),\;\; u(x,b) = 0\;\;(0 < x < a)$$

Q5) a) Let $C_n (n = 1,2,3,.......)$ be the Fourier constants for a function f in $C_p(a,b)$ with respect to an orthonormal set $\{\phi_n(x)\}$ $(n = 1,2,3........)$ in that space. Then prove that all possible linear combinations of the functions $\phi_1(x), \phi_2(x) --- \phi_N(x)$ the combinations

$C_1\phi_1(x) + C_2\phi_2(x) + ----- + C_N\phi_N(x)$ is the best approximation in the mean to $f(x)$ on the fundamental interval $a < x < b$. **[5]**

b) Find the eigenvalues and normalized eigenfunctions of Strum-Liouville problem. **[3]**

$$X'' + \lambda X = 0 \quad X'(0) = 0 \quad X(C) = 0$$

c) If $L = x$ and $m = \frac{\partial}{\partial x}$ are linear operators on $C_p(a,b)$, then show that the product LM and ML are not always the same. **[2]**

Q6) a) If λ_m and λ_n are distict eigenvalues of the Sturm-Liouville problem $\left[r(x)X'(x)\right]' + \left[q(x) + \lambda p(x)\right]X(x) = 0 (a < x < b)$ under the condition $a_1 X(a) + a_2 X^1(a) = 0$, $b_1 X(b) + b_2 X^1(b) = 0$, then prove that corresponding eigenfunctions $X_m(x)$ and $X_n(x)$ are orthogonal with respect to weight function $p(x)$ on the interval $a < x < b$. **[4]**

b) Solve the boundary value problem

$$u_{xx}(x,y) + u_{yy}(x,y) = 0 \quad (0 < x < \pi, y > 0)$$
$$u_x(0,y) = 0, u(\pi,y) = 0 \quad (y > 0)$$
$$-K\,uy(x,0) = f(x) \quad (0 < x < \pi)$$

where K is positive constant. **[4]**

c) If $\phi_0(x) = \frac{1}{\sqrt{2\pi}}, \phi_{2n-1}(x) = \frac{1}{\sqrt{\pi}}\cos nx, \phi_{2n} = \frac{1}{\sqrt{\pi}}\sin nx$, then show that the set $\{\phi_n(x)\}(n = 0,1,2,.......)$ is orthonormal on the interval $-\pi < x < \pi$. **[2]**

Q7) a) Solve the Bessel's differential equation $x^2 y'' + xy' + \left(x^2 - n^2\right)y = 0$. **[5]**

b) Derive the recurrence relation $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$. **[3]**

c) Derive the property $\overline{|(n+1)} = n\overline{|(n)}$ of the gamma function. **[2]**

Q8) a) Prove that the eigen values and corresponding eigen functions of the singular sturm-Liouville problem $\left[\left(1 - x^2\right) X'(x)\right]' + \lambda\ X'(x) = 0\ (-1 < x < 1)$ are $\lambda_n = n(n+1)$ and $X_n(x) = p_n(x)\ (n = 0,1,2---)$ where $p_n(x)$ are Legendre polynomials. Also prove that the set $\{p_n(x)\}(n = 0,1,2----)$ is orthogonal on the interval $-1 < x < 1$ with weight function unity. **[5]**

b) Obtain the integration formula

$$\int_{-1}^{1} p_n(x)dx = \frac{1}{(2n+1)}\left[p_{n-1}(a) - p_{n+1}(a)\right](n = 1,2,3----)$$

c) If $p_n(x)$ is the Legendre's polynomial of degree *n*, then show that $p_n(-x) = (-1)^n p_n(x)\ (n = 0,1,2,.....)$ **[2]**

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