IITJEE–2004 Mains Paper

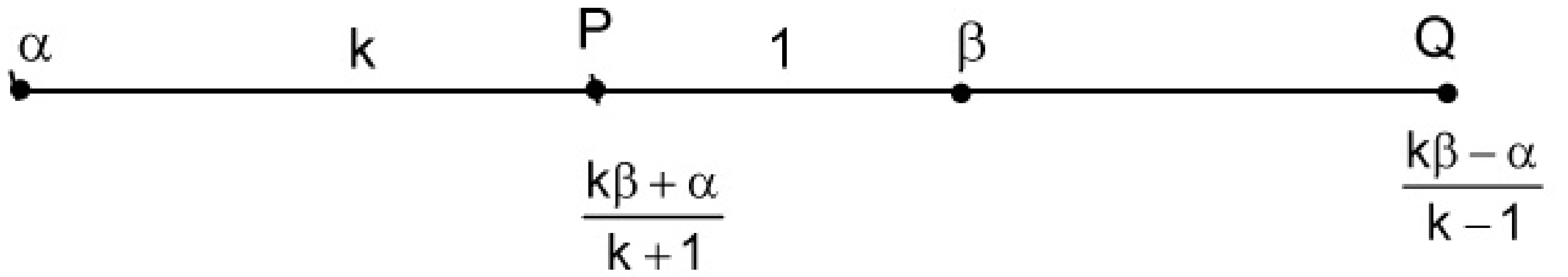
Mathematics

Time: 2 hours

Note: Question number 1 to 10 carries ***2 marks*** each and 11 to 20 carries ***4 marks*** each.

1. Find the centre and radius of the circle formed by all the points represented by z = x + iy satisfying the relation $\frac{|z-\alpha|}{|z-\beta|} = k$ $(k \neq 1)$ where α and β are constant complex numbers given by $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$.

Sol.



Centre is the mid-point of points dividing the join of α and β in the ratio k : 1 internally and externally.

i.e. $z = \frac{1}{2}\left(\frac{k\beta+\alpha}{k+1} + \frac{k\beta-\alpha}{k-1}\right) = \frac{\alpha - k^2\beta}{1-k^2}$

radius = $\left|\frac{\alpha - k^2\beta}{1-k^2} - \frac{k\beta+\alpha}{1+k}\right| = \left|\frac{k(\alpha-\beta)}{1-k^2}\right|$.

Alternative:

We have $\frac{|z-\alpha|}{|z-\beta|} = k$

so that $(z-\alpha)(\bar{z}-\bar{\alpha}) = k^2(z-\beta)(\bar{z}-\bar{\beta})$

or $z\bar{z} - \alpha\bar{z} - \bar{\alpha}z + \alpha\bar{\alpha} = k^2(z\bar{z} - \beta\bar{z} - \bar{\beta}z + \beta\bar{\beta})$

or $z\bar{z}\left(1-k^2\right) - \left(\alpha - \kappa^2\beta\right)\bar{z} - \left(\bar{\alpha} - \kappa^2\bar{\beta}\right)z + \alpha\bar{\alpha} - k^2\beta\bar{\beta} = 0$

or $z\bar{z} - \frac{\left(\alpha - k^2\beta\right)}{1-k^2}\bar{z} - \frac{\left(\bar{\alpha} - k^2\bar{\beta}\right)}{1-k^2}z + \frac{\alpha\bar{\alpha} - k^2\beta\bar{\beta}}{1-k^2} = 0$

which represents a circle with centre $\frac{\alpha - k^2\beta}{1-k^2}$ and radius $\sqrt{\frac{\left(\alpha - k^2\beta\right)\left(\bar{\alpha} - k^2\bar{\beta}\right)}{\left(1-k^2\right)^2} - \frac{\alpha\bar{\alpha} - k^2\beta\bar{\beta}}{\left(1-k^2\right)}} = \left|\frac{k(\alpha-\beta)}{1-k^2}\right|$.

2. $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four distinct vectors satisfying the conditions $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$, then prove that $\vec{a}\cdot\vec{b} + \vec{c}\cdot\vec{d} \neq \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{d}$.

Sol. Given that $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$

$\Rightarrow \vec{a}\times\left(\vec{b}-\vec{c}\right) = \left(\vec{c}-\vec{b}\right)\times\vec{d} = \vec{d}\times\left(\vec{b}-\vec{c}\right) \Rightarrow \vec{a}-\vec{d} \,||\, \vec{b}-\vec{c}$

$\Rightarrow \left(\vec{a}-\vec{d}\right)\cdot\left(\vec{b}-\vec{c}\right) \neq 0 \Rightarrow \vec{a}\cdot\vec{b} + \vec{d}\cdot\vec{c} \neq \vec{d}\cdot\vec{b} + \vec{a}\cdot\vec{c}$.

3. Using permutation or otherwise prove that $\frac{n^2!}{(n!)^n}$ is an integer, where n is a positive integer.

Sol. Let there be n^2 objects distributed in n groups, each group containing n identical objects. So number of arrangement of these n^2 objects are $\frac{n^2!}{(n!)^n}$ and number of arrangements has to be an integer.

Hence $\frac{n^2}{(n!)^n}$ is an integer.

4. If M is a 3 × 3 matrix, where $M^TM = I$ and det (M) = 1, then prove that det (M – I) = 0.

Sol. $(M-I)^T = M^T - I = M^T - M^TM = M^T(I-M)$
$\Rightarrow |(M-I)^T| = |M-I| = |M^T|\,|I-M| = |I-M| \Rightarrow |M-I| = 0.$

Alternate: $\det(M-I) = \det(M-I)\det(M^T) = \det(MM^T - M^T)$
$= \det(I - M^T) = -\det(M^T - I) = -\det(M-I)^T = -\det(M-I) \Rightarrow \det(M-I) = 0.$

5. If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cdot \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\,d\theta$ then find $\frac{dy}{dx}$ at $x = \pi$.

Sol. $y = \int_{\pi^2/16}^{x^2} \frac{\cos x \cdot \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\,d\theta = \cos x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\,d\theta$

so that $\frac{dy}{dx} = -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\,d\theta + \frac{2x\cos x \cdot \cos x}{1+\sin^2 x}$

Hence, at $x = \pi$, $\frac{dy}{dx} = 0 + \frac{2\pi(-1)(-1)}{1+0} = 2\pi$.

6. T is a parallelopiped in which A, B, C and D are vertices of one face. And the face just above it has corresponding vertices A′, B′, C′, D′. T is now compressed to S with face ABCD remaining same and A′, B′, C′, D′ shifted to A″, B″, C″, D″ in S. The volume of parallelopiped S is reduced to 90% of T. Prove that locus of A″ is a plane.

Sol. Let the equation of the plane ABCD be ax + by + cz + d = 0, the point A″ be (α, β, γ) and the height of the parallelopiped ABCD be h.

$\Rightarrow \frac{|a\alpha + b\beta + c\gamma + d|}{\sqrt{a^2+b^2+c^2}} = 0.9\,h. \Rightarrow a\alpha + b\beta + c\gamma + d = \pm 0.9\,h\sqrt{a^2+b^2+c^2}$

⇒ the locus of A″ is a plane parallel to the plane ABCD.

7. If $f : [-1, 1] \to R$ and $f'(0) = \lim_{n\to\infty} nf\left(\frac{1}{n}\right)$ and f (0) = 0. Find the value of $\lim_{n\to\infty} \frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right) - n$.

Given that $0 < \left|\lim_{n\to\infty} \cos^{-1}\left(\frac{1}{n}\right)\right| < \frac{\pi}{2}$.

Sol. $\lim_{n\to\infty} \frac{2}{\pi}(n+1)\cos^{-1}\frac{1}{n} - n = \lim_{n\to\infty} n\left[\frac{2}{\pi}\left(1+\frac{1}{n}\right)\cos^{-1}\frac{1}{n} - 1\right]$

$= \lim_{n\to\infty} n\,f\left(\frac{1}{n}\right) = f'(0)$ where $f(x) = \frac{2}{\pi}(1+x)\cos^{-1}x - 1$.

Clearly, f (0) = 0.

Now, $f'(x) = \frac{2}{\pi}\left[(1+x)\frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x\right]$

$\Rightarrow f'(0) = \frac{2}{\pi}\left[-1+\frac{\pi}{2}\right] = \frac{2}{\pi}\left[\frac{\pi-2}{2}\right] = 1-\frac{2}{\pi}$.

8. If $p(x) = 51x^{101} - 2323x^{100} - 45x + 1035$, using Rolle's Theorem, prove that atleast one root lies between $(45^{1/100}, 46)$.

Sol. Let $g(x) = \int p(x)\,dx = \frac{51x^{102}}{102} - \frac{2323x^{101}}{101} - \frac{45x^2}{2} + 1035x + c$

$= \frac{1}{2}x^{102} - 23x^{101} - \frac{45}{2}x^2 + 1035x + c.$

Now $g(45^{1/100}) = \frac{1}{2}(45)^{\frac{102}{100}} - 23(45)^{\frac{101}{100}} - \frac{45}{2}(45)^{\frac{2}{100}} + 1035(45)^{\frac{1}{100}} + c = c$

$g(46) = \frac{(46)^{102}}{2} - 23(46)^{101} - \frac{45}{2}(46)^2 + 1035(46) + c = c$.

So $g'(x) = p(x)$ will have atleast one root in given interval.

9. A plane is parallel to two lines whose direction ratios are (1, 0, –1) and (–1, 1, 0) and it contains the point (1, 1, 1). If it cuts coordinate axis at A, B, C, then find the volume of the tetrahedron OABC.

Sol. Let (l, m, n) be the direction ratios of the normal to the required plane so that $l - n = 0$ and $-l + m = 0$

$\Rightarrow l = m = n$ and hence the equation of the plane containing (1, 1, 1) is $\frac{x}{3}+\frac{y}{3}+\frac{z}{3} = 1$.

Its intercepts with the coordinate axes are A (3, 0, 0); B (0, 3, 0); C (0, 0, 3). Hence the volume of OABC

$= \frac{1}{6}\begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = \frac{27}{6} = \frac{9}{2}$ cubic units.

10. If A and B are two independent events, prove that $P(A \cup B).\ P(A' \cap B') \le P(C)$, where C is an event defined that exactly one of A and B occurs.

Sol. $P(A \cup B).\ P(A')\,P(B') \le (P(A) + P(B))\,P(A')\,P(B')$

$= P(A).\ P(A')\,P(B') + P(B)\,P(A')\,P(B')$

$= P(A)\,P(B')\,(1 - P(A)) + P(B)\,P(A')\,(1 - P(B))$

$\le P(A)\,P(B') + P(B)\,P(A') = P(C)$.

11. A curve passes through (2, 0) and the slope of tangent at point P (x, y) equals $\frac{(x+1)^2 + y - 3}{(x+1)}$. Find the equation of the curve and area enclosed by the curve and the x-axis in the fourth quadrant.

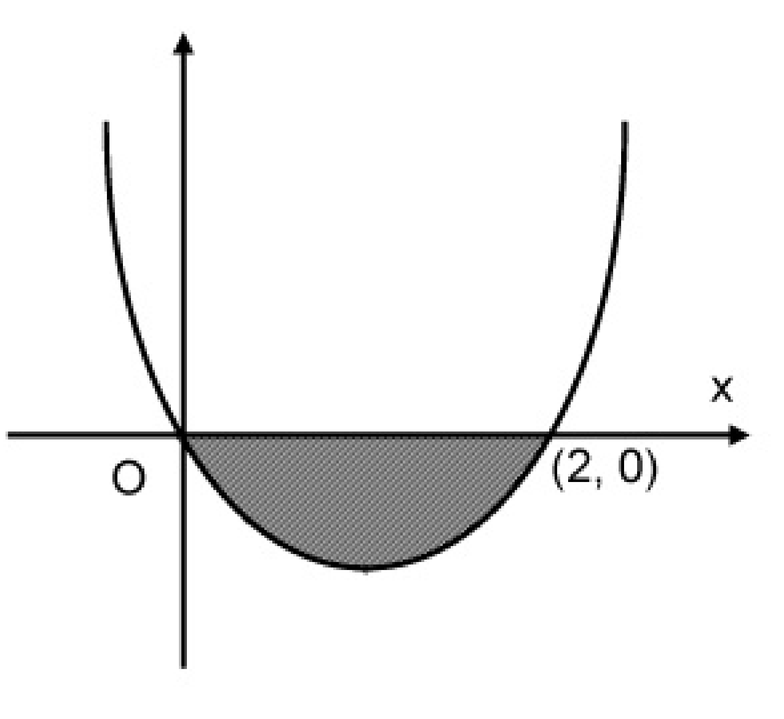
Sol. $\frac{dy}{dx} = \frac{(x+1)^2 + y - 3}{x+1}$

or, $\frac{dy}{dx} = (x+1) + \frac{y-3}{x+1}$

Putting $x + 1 = X$, $y - 3 = Y$

$\frac{dY}{dX} = X + \frac{Y}{X}$

$\frac{dY}{dX} - \frac{Y}{X} = X$



$$I.F = \frac{1}{X} \Rightarrow \frac{1}{X} \cdot Y = X + c$$

$$\frac{y-3}{x+1} = (x+1) + c.$$

It passes through (2, 0) $\Rightarrow c = -4$.

So, $y - 3 = (x+1)^2 - 4(x+1)$

$\Rightarrow y = x^2 - 2x$.

$$\Rightarrow \text{Required area} = \left| \int_0^2 (x^2 - 2x)\, dx \right| = \left| \left[\frac{x^3}{3} - x^2 \right]_0^2 \right| = \frac{4}{3} \text{ sq. units.}$$

12. A circle touches the line $2x + 3y + 1 = 0$ at the point $(1, -1)$ and is orthogonal to the circle which has the line segment having end points $(0, -1)$ and $(-2, 3)$ as the diameter.

Sol. Let the circle with tangent $2x + 3y + 1 = 0$ at $(1, -1)$ be

$(x-1)^2 + (y+1)^2 + \lambda(2x + 3y + 1) = 0$

or $x^2 + y^2 + x(2\lambda - 2) + y(3\lambda + 2) + 2 + \lambda = 0$.

It is orthogonal to $x(x+2) + (y+1)(y-3) = 0$

Or $x^2 + y^2 + 2x - 2y - 3 = 0$

so that $\frac{2(2\lambda-2)}{2} \cdot \left(\frac{2}{2}\right) + \frac{2(3\lambda+2)}{2}\left(\frac{-2}{2}\right) = 2 + \lambda - 3 \Rightarrow \lambda = -\frac{3}{2}$.

Hence the required circle is $2x^2 + 2y^2 - 10x - 5y + 1 = 0$.

13. At any point P on the parabola $y^2 - 2y - 4x + 5 = 0$, a tangent is drawn which meets the directrix at Q. Find the locus of point R which divides QP externally in the ratio $\frac{1}{2} : 1$.

Sol. Any point on the parabola is $P(1 + t^2, 1 + 2t)$. The equation of the tangent at P is $t(y-1) = x - 1 + t^2$ which meets the directrix $x = 0$ at $Q\left(0,\ 1 + t - \frac{1}{t}\right)$. Let R be (h, k).

Since it divides QP externally in the ratio $\frac{1}{2} : 1$, Q is the mid point of RP

$\Rightarrow 0 = \frac{h + 1 + t^2}{2}$ or $t^2 = -(h+1)$

and $1 + t - \frac{1}{t} = \frac{k + 1 + 2t}{2}$ or $t = \frac{2}{1-k}$

So that $\frac{4}{(1-k)^2} + (h+1) = 0$ Or $(k-1)^2(h+1) + 4 = 0$.

Hence locus is $(y-1)^2(x+1) + 4 = 0$.

14. Evaluate $\displaystyle\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$.

Sol. $$I = \int_{-\pi/3}^{\pi/3} \frac{(\pi + 4x^3)\, dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}$$

$$2I = \int_{-\pi/3}^{\pi/3} \frac{2\pi\, dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} = \int_0^{\pi/3} \frac{2\pi\, dx}{2 - \cos\left(x + \frac{\pi}{3}\right)}$$

$$I = \int_{\pi/3}^{2\pi/3} \frac{2\pi\ dt}{2 - \cos t} \Rightarrow I = 2\pi \int_{\pi/3}^{2\pi/3} \frac{\sec^2 \frac{t}{2}\ dt}{1 + 3\tan^2 \frac{t}{2}} = 2\pi \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2\,dt}{1+3t^2} = \frac{4\pi}{3} \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dt}{\left(\frac{1}{\sqrt{3}}\right)^2 + t^2}$$

$$I = \frac{4\pi}{3}\sqrt{3}\left[\tan^{-1}\sqrt{3}t\right]_{1/\sqrt{3}}^{\sqrt{3}} = \frac{4\pi}{\sqrt{3}}\left[\tan^{-1} 3 - \frac{\pi}{4}\right] = \frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right).$$

15. If a, b, c are positive real numbers, then prove that $[(1 + a)(1 + b)(1 + c)]^7 > 7^7 a^4 b^4 c^4$.

Sol. $(1 + a)(1 + b)(1 + c) = 1 + ab + a + b + c + abc + ac + bc$

$\Rightarrow \frac{(1+a)(1+b)(1+c)-1}{7} \geq (ab.\ a.\ b.\ c.\ abc.\ ac.\ bc)^{1/7}$ (using AM ≥ GM)

$\Rightarrow (1 + a)(1 + b)(1 + c) - 1 > 7(a^4 . b^4 . c^4)^{1/7}$

$\Rightarrow (1 + a)(1 + b)(1 + c) > 7(a^4 . b^4 . c^4)^{1/7}$

$\Rightarrow (1 + a)^7 (1 + b)^7 (1 + c)^7 > 7^7 (a^4 . b^4 . c^4)$.

16. $$f(x) = \begin{cases} b\sin^{-1}\left(\frac{x+c}{2}\right), & -\frac{1}{2} < x < 0 \\ \frac{1}{2}, & x = 0 \\ \frac{e^{\frac{a}{2}x} - 1}{x}, & 0 < x < \frac{1}{2} \end{cases}$$

If f (x) is differentiable at x = 0 and $|c| < \frac{1}{2}$ then find the value of 'a' and prove that $64b^2 = (4 - c^2)$.

Sol. $f(0^+) = f(0^-) = f(0)$

Here $f(0^+) = \lim_{x\to\infty} \frac{e^{\frac{ax}{2}} - 1}{x} = \lim_{x\to\infty} \frac{e^{\frac{ax}{2}} - 1}{\frac{ax}{2}} \cdot \frac{a}{2} = \frac{a}{2}$.

$\Rightarrow b\ \sin^{-1}\frac{c}{2} = \frac{a}{2} = \frac{1}{2} \Rightarrow a = 1$.

$$L\ f'(0_-) = \lim_{h\to 0^-} \frac{b\sin^{-1}\frac{(h+c)}{2} - \frac{1}{2}}{h} = \frac{b/2}{\sqrt{1 - \frac{c^2}{4}}}$$

$$R\ f'(0_+) = \lim_{h\to 0^+} \frac{\frac{e^{h/2} - 1}{h} - \frac{1}{2}}{h} = \frac{1}{8}$$

Now $L\ f'(0_-) = R\ f'(0_+) \Rightarrow \frac{\frac{b}{2}}{\sqrt{1 - \frac{c^2}{4}}} = \frac{1}{8}$

$4b = \sqrt{1 - \frac{c^2}{4}} \Rightarrow 16b^2 = \frac{4 - c^2}{4} \Rightarrow 64b^2 = 4 - c^2$.

17. Prove that $\sin x + 2x \geq \frac{3x\cdot(x+1)}{\pi}\ \forall\ x \in \left[0, \frac{\pi}{2}\right]$. (Justify the inequality, if any used).

Sol. Let $f(x) = 3x^2 + (3 - 2\pi)x - \pi \sin x$

$f(0) = 0,\ f\left(\frac{\pi}{2}\right) = -\text{ve}$

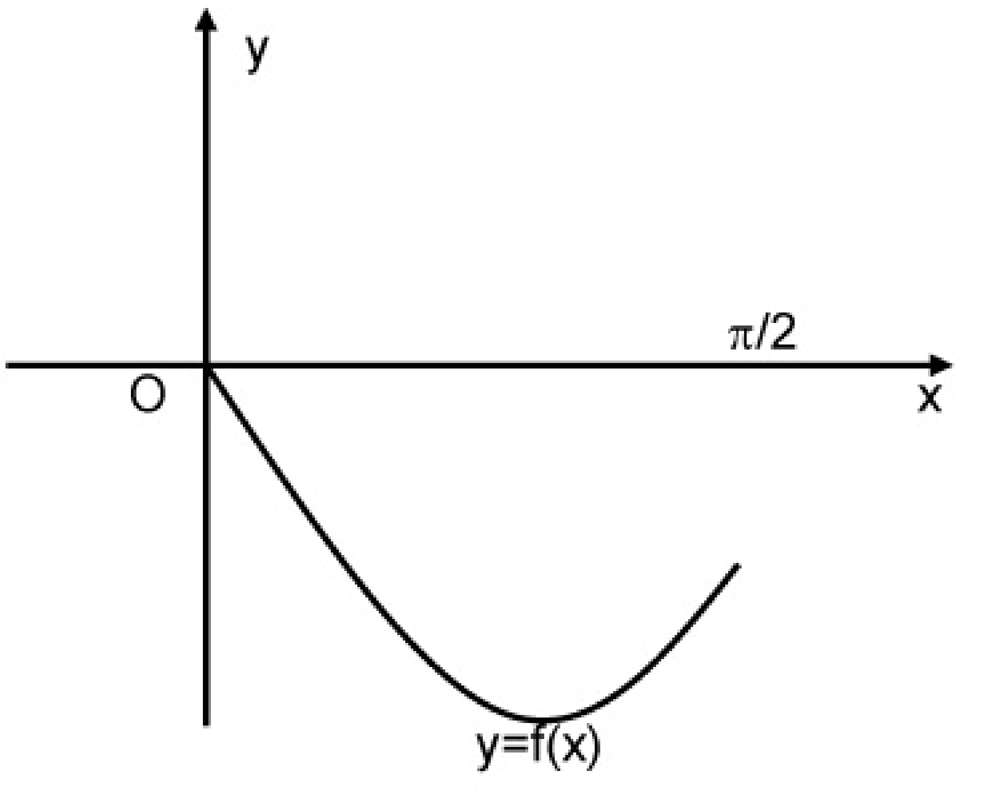
$f'(x) = 6x + 3 - 2\pi - \pi \cos x$

$f''(x) = 6 + \pi \sin x > 0$

$\Rightarrow f'(x)$ is increasing function in $\left[0, \frac{\pi}{2}\right]$

$\Rightarrow$ there is no local maxima of f(x) in $\left[0, \frac{\pi}{2}\right]$

$\Rightarrow$ graph of f(x) always lies below the x-axis in $\left[0, \frac{\pi}{2}\right]$.



$\Rightarrow f(x) \le 0$ in $x \in \left[0, \frac{\pi}{2}\right]$.

$3x^2 + 3x \le 2\pi x + \pi \sin x \Rightarrow \sin x + 2x \ge \frac{3x(x+1)}{\pi}$.

18. $A = \begin{bmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{bmatrix}$, $B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}$, $U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$, $V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$. If there is vector matrix X, such that AX = U has infinitely many solutions, then prove that BX = V cannot have a unique solution. If $afd \ne 0$ then prove that BX = V has no solution.

Sol. AX = U has infinite solutions $\Rightarrow |A| = 0$

$\begin{vmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{vmatrix} = 0 \Rightarrow ab = 1$ or $c = d$

and $|A_1| = \begin{vmatrix} a & 0 & f \\ 1 & c & g \\ 1 & d & h \end{vmatrix} = 0 \Rightarrow g = h$; $|A_2| = \begin{vmatrix} a & f & 1 \\ 1 & g & b \\ 1 & h & b \end{vmatrix} = 0 \Rightarrow g = h$

$|A_3| = \begin{vmatrix} f & 0 & 1 \\ g & c & b \\ h & d & b \end{vmatrix} = 0 \Rightarrow g = h,\ c = d \Rightarrow c = d$ and $g = h$

BX = V

$|B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0$ (since C_2 and C_3 are equal) $\Rightarrow$ BX = V has no unique solution.

and $|B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0$ (since $c = d$, $g = h$)

$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2cf = a^2df$ (since $c = d$)

$$|B_3| = \begin{vmatrix} a & 1 & a^2 \\ 0 & d & 0 \\ f & g & 0 \end{vmatrix} = a^2df$$

since if $adf \neq 0$ then $|B_2| = |B_3| \neq 0$. Hence no solution exist.

19. A bag contains 12 red balls and 6 white balls. Six balls are drawn one by one without replacement of which atleast 4 balls are white. Find the probability that in the next two draws exactly one white ball is drawn. (leave the answer in terms of ${}^nC_r$).

Sol. Let P(A) be the probability that atleast 4 white balls have been drawn.
$P(A_1)$ be the probability that exactly 4 white balls have been drawn.
$P(A_2)$ be the probability that exactly 5 white balls have been drawn.
$P(A_3)$ be the probability that exactly 6 white balls have been drawn.
P(B) be the probability that exactly 1 white ball is drawn from two draws.

$$P(B/A) = \frac{\sum_{i=1}^{3} P(A_i)P(B/A_i)}{\sum_{i=1}^{3} P(A_i)} = \frac{\frac{{}^{12}C_2\,{}^6C_4}{{}^{18}C_6}\cdot\frac{{}^{10}C_1\,{}^2C_1}{{}^{12}C_2} + \frac{{}^{12}C_1\,{}^6C_5}{{}^{18}C_6}\cdot\frac{{}^{11}C_1\,{}^1C_1}{{}^{12}C_2}}{\frac{{}^{12}C_2\,{}^6C_4}{{}^{18}C_6} + \frac{{}^{12}C_1\,{}^6C_5}{{}^{18}C_6} + \frac{{}^{12}C_0\,{}^6C_6}{{}^{18}C_6}}$$

$$= \frac{{}^{12}C_2\,{}^6C_4\,{}^{10}C_1\,{}^2C_1 + {}^{12}C_1\,{}^6C_5\,{}^{11}C_1\,{}^1C_1}{{}^{12}C_2\left({}^{12}C_2\,{}^6C_4 + {}^{12}C_1\,{}^6C_5 + {}^{12}C_0\,{}^6C_6\right)}$$

20. Two planes P_1 and P_2 pass through origin. Two lines L_1 and L_2 also passing through origin are such that L_1 lies on P_1 but not on P_2, L_2 lies on P_2 but not on P_1. A, B, C are three points other than origin, then prove that the permutation [A′, B′, C′] of [A, B, C] exists such that
(i). A lies on L_1, B lies on P_1 not on L_1, C does not lie on P_1.
(ii). A′ lies on L_2, B′ lies on P_2 not on L_2, C′ does not lie on P_2.

Sol. A corresponds to one of A′, B′, C′ and
B corresponds to one of the remaining of A′, B′, C′ and
C corresponds to third of A′, B′, C′.
Hence six such permutations are possible
eg One of the permutations may A ≡ A′; B ≡ B′, C ≡ C′
From the given conditions:
A lies on L_1.
B lies on the line of intersection of P_1 and P_2
and 'C' lies on the line L_2 on the plane P_2.
Now, A′ lies on L_2 ≡ C.
B′ lies on the line of intersection of P_1 and P_2 ≡ B
C′ lie on L_1 on plane P_1 ≡ A.
Hence there exist a particular set [A′, B′, C′] which is the permutation of [A, B, C] such that both (i) and (ii) is satisfied. Here [A′, B′, C′] ≡ [CBA].