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MDE/M-15 **4031**

COMPLEX ANALYSIS

Paper–IV (MM-404)

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *five* questions in all, selecting at least *one* question from each section.

SECTION–I

1. (a) Write notes on the following :

(i) Piecewise smooth arc.

(ii) Simply and Multiply connected region.

(iii) Complex line integral. 8

(b) State and prove Cauchy-Goursat lemma. 8

2. (a) Let $f(z)$ be analytic within and on a positively oriented simple closed contour C and z_0 is any point lying in it, then show that $f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$. 8

(b) State and prove Fundamental theorem of Algebra. Hence show that every polynomial of degree n ($n \geq 1$) has exactly n-roots. 8

3. (a) If a function f is analytic and not constant in a domain D then $|f(z)|$ has no maximum value in D. 8

(b) If $f(z)$ is analytic in a circular domain D, then for every $z \in D$, $f(z)$ can be expressed as

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{\lfloor 2}f''(a) + \cdots$$

$$+ \frac{(z-a)^n}{\lfloor n}f^n(a) + \cdots,$$

where a is the centre of the circular domain. 8

SECTION–II

4. (a) Expand the function $f(z) = \dfrac{1}{z^2 - 3z + 2}$ as Laurent's series in the region (i) $0 < |z| < 1$, (ii) $1 < |z| < 2$, and (iii) $|z| > 2$. 8

(b) State and prove Argument principle. 8

5. (a) Using Contour integration, show that

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}.$$ 8

(b) Show that a bilinear transformation maps inverse points w.r.t. a circle or straight line onto inverse points w.r.t. the image circle and the image line. 8

SECTION–III

6. (a) If $\{f_n\}$ is a sequence in H(G) and f belongs to C(G, C) such that $f_n \to f$ then show that f is analytic and $f_n^{(K)} \to f^{(K)}$ for each integer $K \geq 1$. 8

(b) Define Gamma function, and derive the Legendre duplication formula. 8

7. (a) Derive the Riemann's function equation,

$$\xi(1-z) = 2^{1-z}\pi^{-z}\cos\frac{1}{2}\pi z \lceil z\, \xi(z).$$ 8

(b) Define Riemann Zeta function, $\xi(z)$ and show that for $\mathrm{Re}(z) > 1$,

$$\xi(z)\lceil z = \int_0^\infty \left(e^t - 1\right)^{-1} t^{z-1}\, dt.$$ 8

8. (a) Show that the unit circle is a natural boundary of the function

$$f(z) = \sum_{n=0}^{\infty} z^{\lfloor n}.$$ 8

(b) Using Mittag Leffler's theorem, prove that

$$\operatorname{cosec} z = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2 - z^2}.$$ 8

SECTION–IV

9. (a) State and prove Harnack's inequality. 8

(b) State and prove Hadmard's Three circle theorem. 8

10. (a) Using Hadmard's Factorization theorem, prove that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \qquad 8$$

(b) State and prove Montel-Caratheodory theorem. 8
