Mathematics

Time: 2 hours

Note: Question number 1 to 8 carries ***2 marks*** each, 9 to 16 carries ***4 marks*** each and 17 to 18 carries ***6 marks*** each.

Q1. A person goes to office either by car, scooter, bus or train probability of which being $\frac{1}{7}, \frac{3}{7}, \frac{2}{7}$ and $\frac{1}{7}$ respectively. Probability that he reaches office late, if he takes car, scooter, bus or train is $\frac{2}{9}, \frac{1}{9}, \frac{4}{9}$ and $\frac{1}{9}$ respectively. Given that he reached office in time, then what is the probability that he travelled by a car.

Sol. Let C, S, B, T be the events of the person going by car, scooter, bus or train respectively.

Given that $P(C) = \frac{1}{7}$, $P(S) = \frac{3}{7}$, $P(B) = \frac{2}{7}$, $P(T) = \frac{1}{7}$

Let $\bar{L}$ be the event of the person reaching the office in time.

$\Rightarrow P\left(\frac{\bar{L}}{C}\right) = \frac{7}{9},\ P\left(\frac{\bar{L}}{S}\right) = \frac{8}{9},\ P\left(\frac{\bar{L}}{B}\right) = \frac{5}{9},\ P\left(\frac{\bar{L}}{T}\right) = \frac{8}{9}$

$$\Rightarrow P\left(\frac{C}{\bar{L}}\right) = \frac{P\left(\frac{\bar{L}}{C}\right)P(C)}{P(\bar{L})} = \frac{\frac{1}{7}\times\frac{7}{9}}{\frac{1}{7}\times\frac{7}{9}+\frac{3}{7}\times\frac{8}{9}+\frac{2}{7}\times\frac{5}{9}+\frac{8}{9}\times\frac{1}{7}} = \frac{1}{7}.$$

Q2. Find the range of values of t for which $2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1}$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Sol. Let y = 2 sin t

so, $y = \frac{1-2x+5x^2}{3x^2-2x-1}$

$\Rightarrow (3y-5)x^2 - 2x(y-1) - (y+1) = 0$

since $x \in R - \left\{1, -\frac{1}{3}\right\}$, so $D \ge 0$

$\Rightarrow y^2 - y - 1 \ge 0$

or $y \ge \frac{1+\sqrt{5}}{2}$ and $y \le \frac{1-\sqrt{5}}{2}$

or $\sin t \ge \frac{1+\sqrt{5}}{4}$ and $\sin t \le \frac{1-\sqrt{5}}{4}$

Hence range of t is $\left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$.

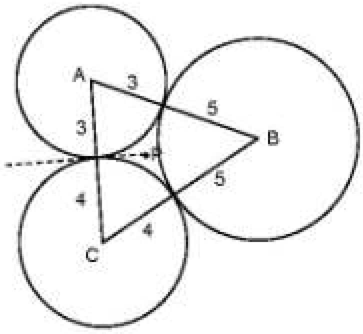
Q3. Circles with radii 3, 4 and 5 touch each other externally if P is the point of intersection of tangents to these circles at their points of contact. Find the distance of P from the points of contact.

Sol. Let A, B, C be the centre of the three circles.
Clearly the point P is the in–centre of the ΔABC, and hence

$$r = \frac{\Delta}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

Now $2s = 7 + 8 + 9 = 24 \Rightarrow s = 12$.

Hence $r = \sqrt{\frac{5.4.3}{12}} = \sqrt{5}$.



Q4. Find the equation of the plane containing the line $2x - y + z - 3 = 0$, $3x + y + z = 5$ and at a distance of $\frac{1}{\sqrt{6}}$ from the point $(2, 1, -1)$.

Sol. Let the equation of plane be $(3\lambda + 2)x + (\lambda - 1)y + (\lambda + 1)z - 5\lambda - 3 = 0$

$$\Rightarrow \left|\frac{6\lambda + 4 + \lambda - 1 - \lambda - 1 - 5\lambda - 3}{\sqrt{(3\lambda+2)^2 + (\lambda-1)^2 + (\lambda+1)^2}}\right| = \frac{1}{\sqrt{6}}$$

$\Rightarrow 6(\lambda - 1)^2 = 11\lambda^2 + 12\lambda + 6 \Rightarrow \lambda = 0, -\frac{24}{5}$.

$\Rightarrow$ The planes are $2x - y + z - 3 = 0$ and $62x + 29y + 19z - 105 = 0$.

Q5. If $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve $y = f(x)$ at the point (1, 2).

Sol. $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$

$$\Rightarrow \lim_{x_1 \to x_2}\left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| < \lim_{x_1 \to x_2} |x_1 - x_2| \Rightarrow |f'(x)| < \delta \Rightarrow f'(x) = 0.$$

Hence f (x) is a constant function and P (1, 2) lies on the curve.
$\Rightarrow f(x) = 2$ is the curve.
Hence the equation of tangent is $y - 2 = 0$.

Q6. If total number of runs scored in n matches is $\left(\frac{n+1}{4}\right)(2^{n+1} - n - 2)$ where $n > 1$, and the runs scored in the k^{th} match are given by $k.2^{n+1-k}$, where $1 \le k \le n$. Find n.

Sol. Let $S_n = \sum_{k=1}^{n} k.2^{n+1-k} = 2^{n+1}\sum_{k=1}^{n} k.2^{-k} = 2^{n+1}.2\left[1 - \frac{1}{2^n} - \frac{n}{2^{n+1}}\right]$ (sum of the A.G.P.)

$= 2[2^{n+1} - 2 - n]$

$\Rightarrow \frac{n+1}{4} = 2 \Rightarrow n = 7$.

Q7. The area of the triangle formed by the intersection of a line parallel to x-axis and passing through P(h, k) with the lines $y = x$ and $x + y = 2$ is $4h^2$. Find the locus of the point P.

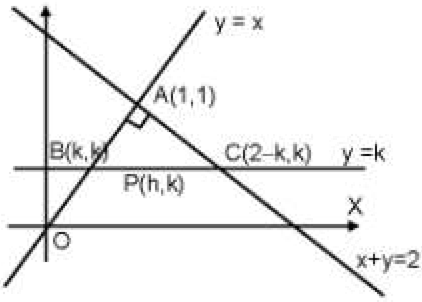
Sol. Area of triangle $= \frac{1}{2}.AB.AC = 4h^2$

and $AB = \sqrt{2}\,|k - 1| = AC$

$\Rightarrow 4h^2 = \frac{1}{2}.2.(k-1)^2$

$\Rightarrow k - 1 = \pm 2h$.

$\Rightarrow$ locus is $y = 2x + 1$, $y = -2x + 1$.



Q8. Evaluate $\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$.

Sol. $I=\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$

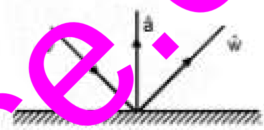
$= 6\int_0^{\pi/2} e^{\cos x}\sin x\cos\left(\frac{1}{2}\cos x\right)dx \qquad \left(\because \int_0^{2a} f(x)\,dx=\begin{cases}0, & \text{if } f(2a-x)=-f(x)\\ 2\int_0^a f(x)\,dx, & \text{if } f(2a-x)=f(x)\end{cases}\right)$

Let cos x = t

$I=6\int_0^1 e^t\cos\left(\frac{t}{2}\right)dt$

$=\frac{24}{5}\left(e\cos\left(\frac{1}{2}\right)+\frac{e}{2}\sin\left(\frac{1}{2}\right)-1\right)$.

Q9. Incident ray is along the unit vector $\hat{v}$ and the reflected ray is along the unit vector $\hat{w}$. The normal is along unit vector $\hat{a}$ outwards. Express $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$.



Sol. $\hat{v}$ is unit vector along the incident ray and $\hat{w}$ is the unit vector along the reflected ray. Hence $\hat{a}$ is a unit vector along the external bisector of $\hat{v}$ and $\hat{w}$. Hence

$\hat{w}-\hat{v}=\lambda\hat{a}$

$\Rightarrow 1+1-\hat{w}\cdot\hat{v}=\lambda^2$

or $2-2\cos 2\theta=\lambda^2$

or $\lambda=2\sin\theta$

where 2θ is the angle between $\hat{v}$ and $\hat{w}$

Hence $\hat{w}-\hat{v}=2\sin\theta\hat{a}=2\cos(90^0-\theta)\hat{a}=-(2\hat{a}\cdot\hat{v})\hat{a}$

$\Rightarrow \hat{w}=\hat{v}-2(\hat{a}\cdot\hat{v})\hat{a}$.

Q10. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ to the circle $x^2+y^2=9$. Find the locus of mid–point of the chord of contact.

Sol. Any point on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ is (3 secθ, 2 tanθ).

Chord of contact of the circle $x^2+y^2=9$ with respect to the point (3 sec θ, 2tan θ) is

$3\sec\theta.x+2\tan\theta.y=9$ ....(1)

Let (x_1, y_1) be the mid–point of the chord of contact.

⇒ equation of chord in mid–point form is $xx_1+yy_1=x_1^2+y_1^2$ ....(2)

Since (1) and (2) represent the same line,

$\frac{3\sec\theta}{x_1}=\frac{2\tan\theta}{y_1}=\frac{9}{x_1^2+y_1^2}$

$\Rightarrow \sec\theta=\frac{9x_1}{3\left(x_1^2+y_1^2\right)},\quad \tan\theta=\frac{9y_1}{2\left(x_1^2+y_1^2\right)}$

Hence $\frac{81x_1^2}{9\left(x_1^2+y_1^2\right)^2}-\frac{81y_1^2}{4\left(x_1^2+y_1^2\right)^2}=1$

⇒ the required locus is $\frac{x^2}{9}-\frac{y^2}{4}=\left(\frac{x^2+y^2}{9}\right)^2$.

Q11. Find the equation of the common tangent in 1[st] quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$. Also find the length of the intercept of the tangent between the coordinate axes.

Sol. Let the equations of tangents to the given circle and the ellipse respectively be

$y = mx + 4\sqrt{1+m^2}$

and $y = mx + \sqrt{25m^2+4}$

Since both of these represent the same common tangent,

$4\sqrt{1+m^2} = \sqrt{25m^2+4}$

$\Rightarrow 16(1 + m^2) = 25m^2 + 4$

$\Rightarrow m = \pm\frac{2}{\sqrt{3}}$

The tangent is at a point in the first quadrant $\Rightarrow m < 0$.

$\Rightarrow m = -\frac{2}{\sqrt{3}}$, so that the equation of the common tangent is

$y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}$.

It meets the coordinate axes at $A\left(2\sqrt{7}, 0\right)$ and $B\left(0, 4\sqrt{\frac{7}{3}}\right)$

$\Rightarrow AB = \frac{14}{\sqrt{3}}$.

Q12. If length of tangent at any point on the curve y = f(x) intercepted between the point and the x–axis is of length 1. Find the equation of the curve.

Sol. Length of tangent = $\left|y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\right| = 1 \Rightarrow y^2\left[1+\left(\frac{dx}{dy}\right)^2\right]$

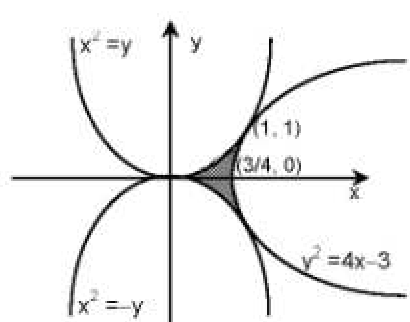
$\Rightarrow \frac{dy}{dx} = \pm\frac{y}{\sqrt{1-y^2}} \Rightarrow \int\frac{\sqrt{1-y^2}}{y}dy = \pm x + c$.

Writing y = sin θ, dy = cos θ dθ and integrating, we get the equation of the curve as

$\sqrt{1-y^2} + \ln\left|\frac{1-\sqrt{1-y^2}}{y}\right| = \pm x + c$.

Q13. Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$.

Sol. The region bounded by the given curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$ is symmetrical about the x–axis. The parabolas $x^2 = y$ and $y^2 = 4x - 3$ touch at the point (1, 1). Moreover the vertex of the curve $y^2 = 4x - 3$ is at $\left(\frac{3}{4}, 0\right)$.



Hence the area of the region

$= 2\left[\int_0^1 x^2dx - \int_{3/4}^{1}\sqrt{4x-3}\,dx\right]$

$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \frac{1}{6}\left((4x-3)^{3/2}\right)_{3/4}^1\right] = 2\left[\frac{1}{3}-\frac{1}{6}\right] = \frac{1}{3}$ sq. units.

Q14. If one of the vertices of the square circumscribing the circle $|z-1| = \sqrt{2}$ is $2+\sqrt{3}\,i$. Find the other vertices of square.

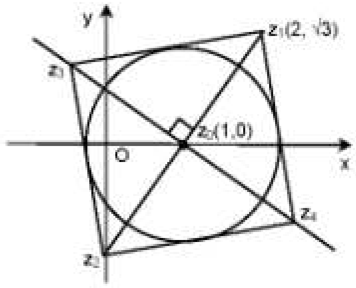
Sol. Since centre of circle i.e. (1, 0) is also the mid–point of diagonals of square

$$\Rightarrow \frac{z_1+z_2}{2} = z_0 \Rightarrow z_2 = -\sqrt{3}i$$

and $\dfrac{z_3-1}{z_1-1} = e^{\pm i\pi/2}$

$\Rightarrow$ other vertices are

$z_3, z_4 = (1-\sqrt{3})+i$ and $(1+\sqrt{3})-i$.



Q15. If f (x – y) = f (x). g (y) – f (y). g (x) and g (x – y) = g (x). g (y) + f (x). f (y) for all x, y ∈ R. If right hand derivative at x = 0 exists for f (x). Find derivative of g (x) at x = 0.

Sol. $f(x-y) = f(x)\,g(y) - f(y)\,g(x)$ .... (1)

Put x = y in (1), we get

$f(0) = 0$

put y = 0 in (1), we get

$g(0) = 1$.

Now, $f'(0^+) = \lim\limits_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{f(0)g(-h)-g(0)f(-h)-f(0)}{h}$

$= \lim\limits_{h\to 0^+} \dfrac{f(-h)}{-h}$ $(\because f(0) = 0)$

$= \lim\limits_{h\to 0^+} \dfrac{f(0-h)-f(0)}{-h}$

$= f'(0^-)$.

Hence f (x) is differentiable at x = 0.

Put y = x in g (x – y) = g (x). g (y) + f (x). f (y).

Also $f^2(x) + g^2(x) = 1$

$\Rightarrow g^2(x) = 1 - f^2(x)$

$\Rightarrow 2g'(0)\,g(0) = -2f(0)\,f'(0) = 0 \Rightarrow g'(0) = 0$.

Q16. If p(x) be a polynomial of degree 3 satisfying p(–1) = 10, p(1) = –6 and p(x) has maximum at x = –1 and p′(x) has minima at x = 1. Find the distance between the local maximum and local minimum of the curve.

Sol. Let the polynomial be $P(x) = ax^3 + bx^2 + cx + d$

According to given conditions

$P(-1) = -a + b - c + d = 10$

$P(1) = a + b + c + d = -6$

Also $P'(-1) = 3a - 2b + c = 0$

and $P''(1) = 6a + 2b = 0 \Rightarrow 3a + b = 0$

Solving for a, b, c, d we get

$P(x) = x^3 - 3x^2 - 9x + 5$

$\Rightarrow P'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$

$\Rightarrow$ x = –1 is the point of maximum and x = 3 is the point of minimum.

Hence distance between (–1, 10) and (3, –22) is $4\sqrt{65}$ units.

Q17. f(x) is a differentiable function and g (x) is a double differentiable function such that $|f(x)| \le 1$ and $f'(x) = g(x)$. If $f^2(0) + g^2(0) = 9$. Prove that there exists some $c \in (-3, 3)$ such that $g(c).\,g''(c) < 0$.

Sol. Let us suppose that both g (x) and g″ (x) are positive for all $x \in (-3, 3)$.

Since $f^2(0) + g^2(0) = 9$ and $-1 \le f(x) \le 1$, $2\sqrt{2} \le g(0) \le 3$.

From $f'(x) = g(x)$, we get

$$f(x) = \int_{-3}^{x} g(x)dx + f(-3).$$

Moreover, $g''(x)$ is assumed to be positive

$\Rightarrow$ the curve $y = g(x)$ is open upwards.

If $g(x)$ is decreasing, then for some value of x $\int_{-3}^{x} g(x)dx$ > area of the rectangle $(0 - (-3))2\sqrt{2}$

$\Rightarrow f(x) > 2\sqrt{2} \times 3 - 1$ i.e. $f(x) > 1$ which is a contradiction.

If $g(x)$ is increasing, for some value of x $\int_{-3}^{x} g(x)dx$ > area of the rectangle $(3 - 0))2\sqrt{2}$

$\Rightarrow f(x) > 2\sqrt{2} \times 3 - 1$ i.e. $f(x) > 1$ which is a contradiction.

If $g(x)$ is minimum at $x = 0$, then $\int_{-3}^{x} g(x)dx$ > area of the rectangle $(3 - 0)2\sqrt{2}$

$\Rightarrow f(x) > 2\sqrt{2} \times 6 - 1$ i.e. $f(x) > 1$ which is a contradiction.

Hence $g(x)$ and $g''(x)$ cannot be both positive throughout the interval $(-3, 3)$.

Similarly we can prove that $g(x)$ and $g''(x)$ cannot be both negative throughout the interval $(-3, 3)$.

Hence there is atleast one value of $c \in (-3, 3)$ where $g(x)$ and $g''(x)$ are of opposite sign

$\Rightarrow g(c) \,.\, g''(c) < 0$.

Alternate:

$$\int_0^3 g(x)dx = \int_0^3 f'(x)dx = f(3) - f(0)$$

$$\Rightarrow \left|\int_0^3 g(x)dx\right| < 2 \qquad \ldots\ldots (1)$$

In the same way $\left|\int_{-3}^{0} g(x)dx\right| < 2 \qquad \ldots\ldots (2)$

$$\Rightarrow \left|\int_0^3 g(x)dx\right| + \left|\int_{-3}^{0} g(x)dx\right| < 4 \qquad \ldots\ldots (3)$$

From $(f(0))^2 + (g(0))^2 = 9$

we get

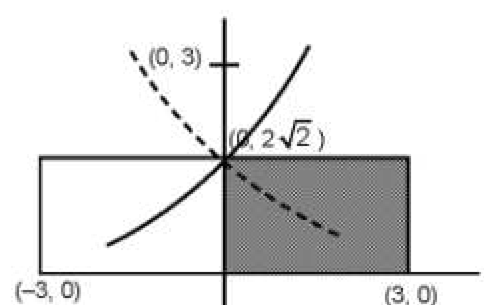
$2\sqrt{2} < g(0) < 3 \qquad \ldots\ldots (4)$

or $-3 < g(0) < -2\sqrt{2} \qquad \ldots\ldots (5)$

Case I: $2\sqrt{2} < g(0) < 3$

Let $g(x)$ is concave upward $\forall$ x $(-3, 3)$ then the area

$$\left|\int_{-3}^{0} g(x)dx\right| + \left|\int_0^3 g(x)dx\right| > 6\sqrt{2}$$



which is a contradiction from equation (3).

$\therefore$ $g(x)$ will be concave downward for some $c \in (-3, 3)$ i.e. $g''(c) < 0 \qquad \ldots\ldots (6)$

also at that point c

$g(c)$ will be greater than $2\sqrt{2}$

$\Rightarrow g(c) > 0 \qquad \ldots\ldots (7)$

From equation (6) and (7)

$g(c) \,.\, g''(c) < 0$ for some $c \in (-3, 3)$.

Case II: $-3 < g(0) < -2\sqrt{2}$

Let g (x) is concave downward $\forall$ x (–3, 3) then the area

$$\left|\int_{-3}^{0} g(x)dx\right| + \left|\int_{0}^{3} g(x)dx\right| > 6\sqrt{2}$$

which is a contradiction from equation (3).

$\therefore$ g (x) will be concave upward for some $c \in (-3, 3)$ i.e. $g''(c) > 0$ ……(8)

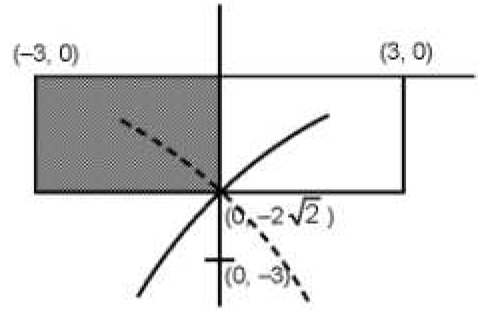
also at that point c

g (c) will be less than $-2\sqrt{2}$

$\Rightarrow g(c) < 0$ ……(9)

From equation (8) and (9)

$g(c) \,.\, g''(c) < 0$ for some $c \in (-3, 3)$.



Q18. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$, f(x) is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of y = f (x) with x-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by f (x) and chord AB.

Sol. Let we have

$4a^2 f(-1) + 4a f(1) + f(2) = 3a^2 + 3a$ … (1)

$4b^2 f(-1) + 4b f(1) + f(2) = 3b^2 + 3b$ … (2)

$4c^2 f(-1) + 4c f(1) + f(2) = 3c^2 + 3c$ … (3)

Consider a quadratic equation

$4x^2 f(-1) + 4x f(1) + f(2) = 3x^2 + 3x$

or $[4f(-1) - 3]x^2 + [4f(1) - 3]x + f(2) = 0$ … (4)

As equation (4) has three roots i.e. x = a, b, c. It is an identity.

$\Rightarrow f(-1) = \frac{3}{4}$, $f(1) = \frac{3}{4}$ and $f(2) = 0$

$\Rightarrow f(x) = \frac{(4 - x^2)}{4}$ … (5)

Let point A be (–2, 0) and B be $(2t, -t^2 + 1)$

Now as AB subtends a right angle at the vertex V (0, 1)

$\frac{1}{2} \times \frac{-t^2}{2t} = -1 \Rightarrow t = 4$

$\Rightarrow B \equiv (8, -15)$

$\therefore$ Area $= \int_{-2}^{8} \left(\frac{4 - x^2}{4} + \frac{3x + 6}{2}\right) dx = \frac{125}{3}$ sq. units.