IITJEE 2009

Paper 1 Code (0)

Maths

21. Let $P(3,2,6)$ be a point in space and Q be a point on the line

$$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-3\hat{i} + \hat{j} + 5\hat{k}).$$

Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is

(A) $\frac{1}{4}$ (B) $-\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $-\frac{1}{8}$

$\boxed{21.}$ $P \equiv (3,2,6) \Rightarrow \bar{p} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

Q $\bar{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-3\hat{i} + \hat{j} + 5\hat{k})$

let pv of Q be $\bar{q} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(-3\hat{i} + \hat{j} + 5\hat{k})$

$\overline{PQ} = \bar{q} - \bar{p} = (-2\hat{i} - 3\hat{j} - 4\hat{k}) + \lambda(-3\hat{i} + \hat{j} + 5\hat{k})$

As $PQ \parallel$ to the plane,

$PQ \perp$ normal of the plane. let $\bar{n}$ be normal of the plane.

$\Rightarrow \overline{PQ} \cdot \bar{n} = 0$

$\Rightarrow \left[(-2-3\lambda)\hat{i} + (-3+\lambda)\hat{j} + (-4+5\lambda)\hat{k}\right] \cdot \left[\hat{i} - 4\hat{j} + 3\hat{k}\right] = 0$

$\Rightarrow -2 - 3\lambda - 4(-3+\lambda) + 3(-4+5\lambda) = 0$

$8\lambda - 2 = 0$ $\lambda = \frac{1}{4}$ $\boxed{A}$

22. Tangents drawn from the point $P(1,8)$ to the circle

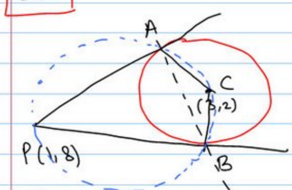
$$x^2+y^2-6x-4y-11=0$$

touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is

(A) $x^2+y^2+4x-6y+19=0$ (B) $x^2+y^2-4x-10y+19=0$

(C) $x^2+y^2-2x+6y-29=0$ (D) $x^2+y^2-6x-4y+19=0$

22.



$S \equiv x^2+y^2-6x-4y-11=0$

Points P, A, C, B are concyclic as $\angle PAC + \angle PBC = \pi$.

$\Rightarrow$ Circumcircle of $\triangle PAB$ will pass through C.

As $\angle PAC$ is $\pi/2$, we can use diametric form. Where PC is the diameter of the circumcircle of $\triangle PAB$.

Using diametric form,

$(x-1)(x-3)+(y-8)(y-2)=0$

$\Rightarrow x^2+y^2-4x-10y+19=0$

B

23. Let f be a non-negative function defined on the interval $[0,1]$. If

$$\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\,dt, \quad 0 \le x \le 1,$$

and $f(0)=0$, then

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$ (B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$ (D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

23

Differentiate wrt x

$$\sqrt{1-f'(x)^2} = f^2(x)$$

As $f(x) \geq 0$, we can take square,

$$1-[f'(x)]^2 = f(x)^2 \quad \text{---(i)}$$

let $f(x) = y \Rightarrow f'(x) = \frac{dy}{dx}$. Replace in (i) to get:

$$1-y^2 = \left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{dy}{dx} = \pm\sqrt{1-y^2}$$

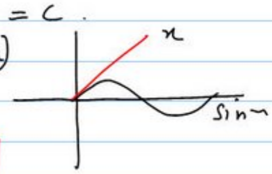
$$\int \frac{dy}{\sqrt{1-y^2}} = \pm \int dx \Rightarrow \sin^{-1} y = \pm x + c$$

$$f(0) = 0 \Rightarrow 0 = c.$$

$\Rightarrow y = \sin x$ (reject $y = -\sin x$)

$\sin x < x \quad \forall x \in (0, \infty)$

$\Rightarrow f\left(\frac{1}{2}\right) < \frac{1}{2}$ & $f\left(\frac{1}{3}\right) < \frac{1}{3}$.



$\boxed{C}$

24. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation

$$z\bar{z}^3 + \bar{z}z^3 = 350$$

is

(A) 48 (B) 32 (C) 40 (D) 80

24

$$(z\bar{z})\,\bar{z}^2 + (\bar{z}z)\,z^2 = 350$$

$$|z|^2\left(\overline{z^2} + (z^2)\right) = 350$$

$$|z|^2\, 2\,\mathrm{Re}(z^2) = 350$$

$$\Rightarrow (x^2+y^2)\,2\,(x^2-y^2) = 350$$

$$\Rightarrow x^4 - y^4 = 175$$

As x and y are integer, we have following cases:

$x = 4, y = 3$ or $x = 4, y = -3$, $x = -4, y = 3$, $x = -4, y = -3$.

$\Rightarrow$ Vertices of the rectangle are:

$(4, 3), (4, -3), (-4, 3), (-4, -3)$

$\Rightarrow$ Area of rectangle $= lb = 8 \times 6 = 48$ $\boxed{A}$

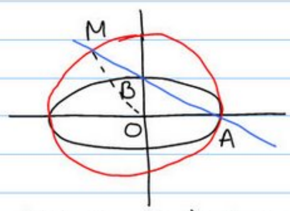
25. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse

$$x^2 + 9y^2 = 9$$

meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

$\boxed{25.}$



Equation of AM $\equiv \frac{x}{a} + \frac{y}{b} = 1$ ---(i)

Homogenise equation of circle (Auxillary) & (i) to get eqn. of pair of lines OA and OM.

$\Rightarrow x^2 + y^2 = a^2\left[\frac{x}{a} + \frac{y}{b}\right]^2 \Rightarrow y^2\left(1 - \frac{a^2}{b^2}\right) - \frac{2a}{b}xy = 0$

Using angle between pair of lines formula,

$$\tan(\angle AOM) = \frac{2\sqrt{\frac{a^2}{b^2} - 0}}{1 - \frac{a^2}{b^2}} = \frac{2ab}{b^2 - a^2} = \frac{2 \times 3 \times 1}{9 - 1} = \frac{3}{4}.$$

Area of $\triangle$ AOM $= \frac{1}{2}$ OA OM $\sin(\angle AOM)$

$$= \frac{1}{2}a^2\frac{3}{5} = \frac{1}{2} \times 9 \times \frac{3}{5} = \frac{27}{10}$$

$\boxed{D}$

26. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are unit vectors such that

$$\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = 1$$

and $\quad \vec{a} \cdot \vec{c} = \frac{1}{2},$

then

(A) $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar

(B) $\vec{b}, \vec{c}, \vec{d}$ are non-coplanar

(C) $\vec{b}, \vec{d}$ are non-parallel

(D) $\vec{a}, \vec{d}$ are parallel and $\vec{b}, \vec{c}$ are parallel

$\boxed{26.}$ $|\bar{a}| = |\bar{b}| = |\bar{c}| = |\bar{d}| = 1$

$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = 1$

$\Rightarrow$ $|\bar{a} \times \bar{b}|\,|\bar{c} \times \bar{d}| \cos\gamma = 1$ where γ is angle between $\bar{a} \times \bar{b}$ and $\bar{c} \times \bar{d}$

$\Rightarrow$ $\sin\alpha \sin\beta \cos\gamma = 1$

where α and β are angles between $\bar{a}$ and $\bar{b}$, $\bar{c}$ and $\bar{d}$ respectively

As $-1 \le \sin\alpha \le 1$, $-1 \le \sin\beta \le 1$, $-1 \le \cos\gamma \le 1$,

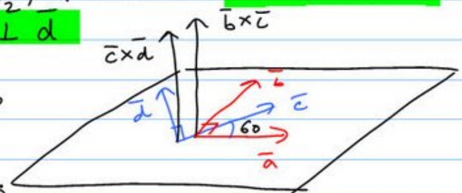
above is only possible for $\sin\alpha = 1$, $\sin\beta = 1$, $\cos\gamma = 1$

$\Rightarrow$ $\alpha = \pi/2$, $\beta = \pi/2$, $\gamma = 0$ $\Rightarrow$ $\bar{a} \times \bar{b} \parallel \bar{c} \times \bar{d}$

$\Rightarrow$ $\bar{a} \perp \bar{b}$, $\bar{c} \perp \bar{d}$

$\bar{a} \cdot \bar{c} = \frac{1}{2}$

$\Rightarrow$ $\cos\theta = \frac{1}{2}$, $\theta = 60°$



From above data, we can make following relationship diagram among vectors.

From figure, $\boxed{\bar{b} \text{ and } \bar{d} \text{ are non-parallel}}$ $\boxed{C}$

All other choices are false.

27. Let $z = \cos\theta + i\sin\theta$. Then the value of

$$\sum_{m=1}^{15} \text{Im}(z^{2m-1})$$

at $\theta = 2°$ is

(A) $\dfrac{1}{\sin 2°}$ (B) $\dfrac{1}{3\sin 2°}$ (C) $\dfrac{1}{2\sin 2°}$ (D) $\dfrac{1}{4\sin 2°}$

$$z = \cos\theta + i\sin\theta = e^{i\theta}$$

$$E = \sum_{m=1}^{15} \mathrm{Im}\left[z^{2m-1}\right] = \sum_{m=1}^{15} \sin(2m-1)\theta$$

$$= \sin\theta + \sin 3\theta + \text{---} \text{ upto 15 terms}$$

Using sum of sine series where angles are in A.P, we get:

$$\frac{\sin \frac{15(2\theta)}{2}}{\sin\left(\frac{2\theta}{2}\right)} \sin\left[15\theta\right] = \frac{\sin^2 15\theta}{\sin\theta} = \frac{\sin^2 30^\circ}{\sin 2^\circ}$$

$$= \frac{1}{4\sin 2^\circ}$$

$\boxed{D}$

Using $\sin a + \sin(a+d) + \sin(a+2d) + \text{---}$ upto n terms

$$= \frac{\sin \frac{nd}{2}}{\sin \frac{d}{2}} \sin\left(\frac{a + a + \overline{n-1}\,d}{2}\right)$$

28. The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is

(A) 55 (B) 66 (C) 77 (D) 88

Using 1, 2, 3 digits

let x_1, x_2, x_3 be the number of times 1, 2, 3 be choosen respectively.

According to given equation, we get

$$x_1 + 2x_2 + 3x_3 = 10 \text{ and } x_1 + x_2 + x_3 = 7$$

We will make cases by inspection.

$(5, 1, 1)$ i.e. 11111, 2, 3 $= \frac{\lfloor 7}{\lfloor 5} = 42$

$(4, 3, 0)$ i.e. 1111, 222 $= \frac{\lfloor 7}{\lfloor 4 \lfloor 3} = 35$

Total numbers $= 77$

$\boxed{C}$

29. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is

(A) $e - 1$

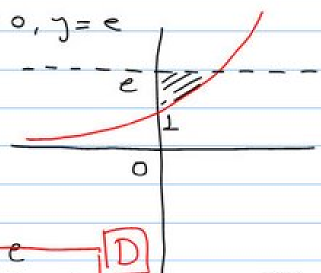
(B) $\int_1^e \ln(e + 1 - y)\,dy$

(C) $e - \int_0^1 e^x\,dx$

(D) $\int_1^e \ln y\,dy$

29.

$y = e^x, \; x = 0, \; y = e$



Area bounded

$= \int_1^e x\,dy$

$= \boxed{\int_1^e \ln y\,dy}$ $\boxed{D}$ --- (i)

$= \ln y \int_1^e y\,dy - \int_1^e dy = y \ln y\Big|_1^e - y\Big|_1^e = e - (e-1) = 1$ --- (ii)

Using (a+b−x) property on (i), we get

Area $= \int_1^e \ln(1+e-y)\,dy$ $\boxed{B}$

Let us check (C) choice.

$e - \int_0^1 e^x\,dx = e - e^x\Big|_0^1 = e - (e-1) = 1$

$\boxed{C}$ using (ii)

30. Let

$$L = \lim_{x \to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}, \quad a > 0.$$

If L is finite, then

(A) $a = 2$ (B) $a = 1$ (C) $L = \frac{1}{64}$ (D) $L = \frac{1}{32}$

30.

$$L = \lim_{x \to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4} \qquad (a > 0)$$

Using LH rule

$$= \lim_{x \to 0} \frac{-\frac{1}{2\sqrt{a^2-x^2}}(-2x) - \frac{x}{2}}{4x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sqrt{a^2-x^2}} - \frac{1}{2}}{4x^2}$$

As Dinominator $\to 0$, Numerator should approach to 0.

$$\Rightarrow \frac{1}{\sqrt{a^2}} - \frac{1}{2} = 0 \qquad \frac{1}{a} = \frac{1}{2}$$

$$\Rightarrow \boxed{a = 2} \qquad \boxed{A}$$

$$L = \lim_{x \to 0} \frac{\frac{1}{\sqrt{4-x^2}} - \frac{1}{2}}{4x^2} = \lim_{x \to 0} \frac{-\frac{1}{2(4-x^2)^{3/2}} \cdot -2x}{8x}$$

$$= \lim_{x \to 0} \frac{1}{8(4-x^2)^{3/2}} = \boxed{\frac{1}{64}} \qquad \boxed{C}$$

31. In a triangle ABC with fixed base BC, the vertex A moves such that

$$\cos B + \cos C = 4\sin^2\frac{A}{2}.$$

If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then

(A) $b + c = 4a$

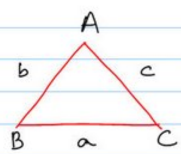
(B) $b + c = 2a$

(C) locus of point A is an ellipse

(D) locus of point A is a pair of straight lines

$\boxed{31}$

$\cos B + \cos C = 4\sin^2\frac{A}{2}$



$2\cos\frac{B+C}{2}\cos\left(\frac{B-C}{2}\right) = 4\sin^2\frac{A}{2}$

$\Rightarrow 2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right) = 4\sin^2\frac{A}{2}$

$\Rightarrow \cos\frac{B-C}{2} = 2\sin\frac{A}{2}$

Multiply both sides by $\cos A/2$ to get:

$\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = \sin A$ $\left[\text{using } \cos\frac{A}{2} = \sin\left(\frac{B+C}{2}\right)\right]$

$\frac{\sin B + \sin C}{2} = \sin A \Rightarrow b + c = 2a$ $\boxed{B}$

$b + c = 2a \Rightarrow AB + AC = 2a$

Above eqn represents ellipse if $2a >$ distance between fixed points

i.e. $2a > a$ which is true.

$\Rightarrow$ Locus is an ellipse. $\boxed{C}$

32. If

$$\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5},$$

then

(A) $\tan^2 x = \frac{2}{3}$

(B) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(C) $\tan^2 x = \frac{1}{3}$

(D) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

$\boxed{32.}$

Let $\sin^2 x = t$

$\Rightarrow \dfrac{t^2}{2} + \dfrac{(1-t)^2}{3} = \dfrac{1}{5}$

$\Rightarrow 3t^2 + 2(t^2 + 1 - 2t) = \dfrac{6}{5}$

$\Rightarrow 25t^2 - 20t + 4 = 0$

$\Rightarrow (5t-2)^2 = 0$

$\Rightarrow \sin^2 x = 2/5 \Rightarrow \tan^2 x = 2/3$ $\boxed{A}$

Consider (B) choice

$LHS = \dfrac{\sin^8 x}{8} + \dfrac{\cos^8 x}{27} = \dfrac{(2/5)^4}{8} + \dfrac{(3/5)^4}{27} = \dfrac{2}{625} + \dfrac{3}{625}$

$= \dfrac{5}{625} = \dfrac{1}{125}$ $\boxed{B}$

Paragraph for Question Nos. 33 to 35

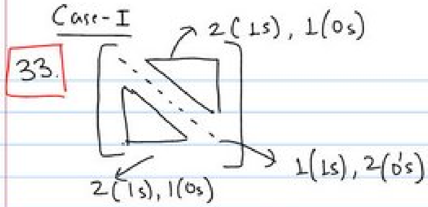
Let $\mathscr{A}$ be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

33. The number of matrices in $\mathscr{A}$ is

(A) 12 (B) 6 (C) 9 (D) 3

Case-I

33.



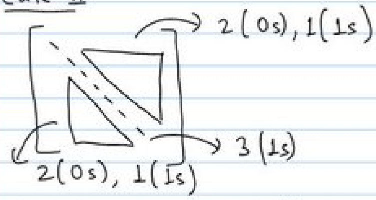
Number of ways to fill upper part = ${}^3C_2 \times {}^1C_1 = 3$

Number of ways to fill lower part = 1 ($\because$ Symmetric)

Number of ways to fill diagonal = ${}^3C_1 \times {}^2C_2 = 3$

Total number of ways = $3 \times 1 \times 3 = 9$

Case-II



Number of ways to fill upper part = ${}^3C_2 \times {}^1C_1 = 3$

Number of ways to fill lower part = 1

Number of ways to fill diagonal = 1

Total ways = $3 \times 1 \times 1 = 3$

Total number of ways combining Case-I, Case-II

$= 9 + 3 = 12$

A

34. The number of matrices A in $\mathscr{A}$ for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution, is

(A) less than 4

(B) at least 4 but less than 7

(C) at least 7 but less than 10

(D) at least 10

34. For unique Solution $D \neq 0$

Following are Ds based on given conditions.

$\overset{D=0}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}}$, $\overset{D=0}{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}$, $\overset{D=0}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}}$, $\overset{D=-1}{\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix}}$, $\overset{D=-1}{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}}$,

$\overset{D=0}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}}$, $\overset{D=0}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}}$, $\overset{D=-1}{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}}$, $\overset{D=-1}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}}$,

$\overset{D=-1}{\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}}$, $\overset{D=0}{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}}$, $\overset{D=-1}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}}$

Number of non-zero determinents $= 6$ $\boxed{\text{B}}$

35. The number of matrices A in $\mathscr{A}$ for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is inconsistent, is

(A) 0 (B) more than 2 (C) 2 (D) 1

$\boxed{35.}$

System is inconsistent if $D = 0$ and either $D_1 \neq 0$, or $D_2 \neq 0$, or $D_3 \neq 0$.

Following are $D = 0$ cases

(1) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ (2) $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (3) $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

(1) $D_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$

$D_2 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$

$D_3 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$,

Infinite Soln.

(2) $D_1 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

$D_1 \neq 0$

No Solution.

(3) $D_1 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$

$D_1 \neq 0$

$\Rightarrow$ No Solution.

(4) $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

$D_1 = 0$; $D_2 \neq 0$, $\Rightarrow$ No Solution.

$\Rightarrow$ There are more than 2 system of equations which are inconsistent.

$\boxed{\text{B}}$

Paragraph for Question Nos. 36 to 38

A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required.

36. The probability that $X = 3$ equals

(A) $\frac{25}{216}$ (B) $\frac{25}{36}$ (C) $\frac{5}{36}$ (D) $\frac{125}{216}$

37. The probability that $X \geq 3$ equals

(A) $\frac{125}{216}$ (B) $\frac{25}{36}$ (C) $\frac{5}{36}$ (D) $\frac{25}{216}$

38. The conditional probability that $X \geq 6$ given $X > 3$ equals

(A) $\frac{125}{216}$ (B) $\frac{25}{216}$ (C) $\frac{5}{36}$ (D) $\frac{25}{36}$

$\boxed{36\text{-}38}$

36. $P(X=3) = P(6 \text{ occurs in } 3 \text{ tosses})$

$= P(\text{1-5 on first throw})\, P(\text{1-5 on second throw}) \times P(6 \text{ on third throw})$

$= \frac{5}{6}\,\frac{5}{6}\,\frac{1}{6} = \frac{25}{216}$ $\boxed{A}$

37. $P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + \cdots \infty$

$= P(\bar{6}\,\bar{6}\,6) + P(\bar{6}\,\bar{6}\,\bar{6}\,6) + P(\bar{6}\,\bar{6}\,\bar{6}\,\bar{6}\,6) + \cdots \infty$

$= \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^3 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \cdots \infty$

$= \dfrac{(5/6)^2 \frac{1}{6}}{1-5/6} = \frac{25}{36}$ $\boxed{B}$

38. $P\left(X \geq 6 / X > 3\right)$

$= \dfrac{P(X=6) + P(X=7) + \cdots \infty}{P(X=4) + P(X=5) + P(X=6) + \cdots \infty}$

$= \dfrac{\left(\frac{5}{6}\right)^5 \frac{1}{6} + \left(\frac{5}{6}\right)^6 \frac{1}{6} + \cdots \infty}{\left(\frac{5}{6}\right)^3 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \cdots \infty} = \dfrac{\dfrac{(5/6)^5 \frac{1}{6}}{1-5/6}}{\dfrac{(5/6)^3 \frac{1}{6}}{1-5/6}}$

$= \dfrac{(5/6)^5}{(5/6)^3} = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$ $\boxed{D}$

39. Match the statements/expressions in **Column I** with the open intervals in **Column II**.

Column I	Column II
(A) Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2\, y' + y = 0$	(p) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(B) Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\, dx$	(q) $\left(0, \frac{\pi}{2}\right)$
(C) Interval in which at least one of the points of local maximum of $\cos^2 x + \sin x$ lies	(r) $\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$
(D) Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s) $\left(0, \frac{\pi}{8}\right)$
	(t) $(-\pi, \pi)$

$\boxed{39}$

A. $(x-3)^2 y' + y = 0$

$\Rightarrow \dfrac{dy}{y} = -\dfrac{dx}{(x-3)^2}$

Integrate to get:

$$\int \frac{dy}{y} = \int \frac{-dx}{(x-3)^2} \Rightarrow \ln|y| = \frac{1}{x-3} + C$$

As $\ln|y|$ can take all values, RHS can take all values except $x = 3$.

$\Rightarrow$ domain is $x \in R - \{3\}$

Matching choices are: $\boxed{p, q, s}$

B. $I = \int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\,dx$

Apply $(a+b-x)$ property,

$$I = \int_1^5 (6-x-1)(6-x-2)(6-x-3)(6-x-4)(6-x-5)\,dx$$

$$= -\int_1^5 (x-5)(x-4)(x-3)(x-2)(x-1)\,dx = -I$$

$\Rightarrow 2I = 0 \Rightarrow I = 0$. Matching choices are $\boxed{p, t}$

C. $f(x) = \cos^2 x + \sin x$

$f'(x) = -\sin 2x + \cos x$

$= -\cos x(2\sin x - 1) = 0$

Critical points: $x = \pi/2, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

$+ \;|\; - \;|\; + \;|\; - \;|\; +$ at $\pi/6$, $\pi/2$, $5\pi/6$, $\frac{3\pi}{2}$

Points of local maximum are $x = \pi/6$, $x = 5\pi/6$

Matching choices are: $\boxed{p, q, r, t}$

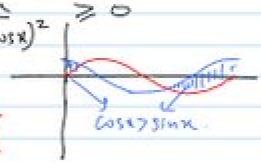
D. $f(x) = \tan^{-1}(\sin x + \cos x)$

$f'(x) = \dfrac{\cos x - \sin x}{1 + (\sin x + \cos x)^2} \geq 0$

$\Rightarrow \cos x \geq \sin x$

$\Rightarrow x \in [0, \pi/4] \cup [\frac{5\pi}{4}, 2\pi]$

Matching choices are $\boxed{s}$



40. Match the conics in **Column I** with the statements/expressions in **Column II**.

Column I	**Column II**
(A) Circle	(p) The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(B) Parabola	(q) Points z in the complex plane satisfying $\lvert z+2\rvert - \lvert z-2\rvert = \pm 3$
(C) Ellipse	(r) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(t) Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \lvert z\rvert^2 + 1$

40.

p $hx + ky = 1$ touches $x^2 + y^2 = 4$

Using condition of tangency i.e. $c^2 = a^2(1+m^2)$, we get:

$$\frac{1}{k^2} = 4\left(1 + \left(-\frac{h}{k}\right)^2\right) \Rightarrow 1 = 4(k^2 + h^2)$$

$\Rightarrow h^2 + k^2 = \frac{1}{4} \Rightarrow$ locus is $x^2 + y^2 = \frac{1}{4}$ A

q $|z - z_1| - |z - z_2| = K$ represents hyperbola if $|z_1 - z_2| > |K|$

Therefore $|z+2| - |z-2| = \pm 3$ represents hyperbola if $4 > 3$ which is true. D

r $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right) \Rightarrow \frac{x^2}{3} = \frac{(1-t^2)^2}{(1+t^2)^2} = \frac{(1+t^2)^2 - 4t^2}{(1+t^2)^2}$

$\Rightarrow \frac{x^2}{3} = 1 - \left(\frac{2t}{1+t^2}\right)^2 \Rightarrow \frac{x^2}{3} = 1 - y^2 \Rightarrow \frac{x^2}{3} + y^2 = 1$. C

s. Eccentricity of parabola $= 1$, hyperbola > 1

ellipse < 1, circle $= 0$

$\Rightarrow$ parabola and hyperbola. B, D

t. $\text{Re}(z+1)^2 = |z|^2 + 1$

let $z = x + iy$. $\Rightarrow (x+1)^2 - y^2 = x^2 + y^2 + 1$

$\Rightarrow 2x - y^2 = y^2 \Rightarrow y^2 = x$ parabola B

Choices

A → p
B → s, t
C → r
D → q, s