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IIT-JEE 2011 Mathematics Paper-I

PART III - MATHEMATICS

SECTION - I (Total Marks : 21)

(Single Correct Answer Type)

This section contains 7 multiple choice questions. Each question has four choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

47. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then

(A) $P \subset Q$ and $Q - P \neq \varnothing$ (B) $Q \not\subset P$ (C) $P \not\subset Q$ (D) $P = Q$

47. **(D)** $P : \sin\theta - \cos\theta = \sqrt{2}\cos\theta \Rightarrow \tan\theta = 1 + \sqrt{2}$

$Q : \sin\theta + \cos\theta = \sqrt{2}\sin\theta \Rightarrow \tan\theta = \quad = \sqrt{2} + 1$

$\therefore \quad P = Q.$

48. Let the straight line $x = b$ divide the area enclsoed by $y = (1 - x)^2$, $y = 0$, and $x = 0$ into two parts R_1 $(0 \le x \le b)$ and R_2 $(0 \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals

$\frac{a_{10} - 2a_8}{2a_9} = \frac{(\alpha^{10} - \beta^{10}) - 2(\alpha^8 - \beta^8)}{2(\alpha^9 - \beta^9)} = \frac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2}{2(\alpha^9 - \beta^9)}$

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

48. **(B)** $R_1 - R_2 = \frac{1}{4} \Rightarrow \int_0^b (1-x)^2 dx - \int_b^1 (1-x)^2 dx = \frac{1}{4} \Rightarrow \frac{1}{3} - \frac{2(1-b)^3}{3} = \frac{1}{4}$

$\Rightarrow (1-b)^3 = \frac{1}{8} \Rightarrow b = \frac{1}{2}.$

49. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \ge 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is

(A) 1 (B) 2 (C) 3 (D) 4

49. **(C)** Since $\alpha^2 - 6\alpha - 2 = 0 \Rightarrow \alpha^2 - 2 = 6\alpha$
and $\beta^2 - 6\beta - 2 = 0 \Rightarrow \beta^2 - 2 = 6\beta$

.

50. A straight line L through the point (3, – 2) is inclined at an angle 60° to the line $\sqrt{3}x + y = 1$. If L also intersects the x-axis, then the equation of L is

(A) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$ (B) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$

(C) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$ (D) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

50. **(B)** If required line has slope = m

$$\Rightarrow \quad \left|\frac{m+\sqrt{3}}{1-\sqrt{3}m}\right| = \tan 60^\circ \quad \Rightarrow \quad m = 0, \sqrt{3}$$

Since, the line intersects the x-axis

$\Rightarrow$ equation is, $(y + 2) = \sqrt{3}(x - 3)$ $\Rightarrow$ $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$.

51. Let (x_0, y_0) be the solution of the following equations

$(2x)^{\ln 2} = (3y)^{\ln 3}$

$3^{\ln x} = 2^{\ln y}$

Then x_0 is

(A) 1 / 6 (B) 1 / 3 (C) 1 / 2 (D) 6

51. **(C)** Since $(\ln 2)(\ln 2 + \ln x) = (\ln 3)(\ln 3 + \ln y)$ and $\ln x \,.\, \ln 3 - \ln y \,.\, \ln 2 = 0$

$\Rightarrow$ $(\ln 2)\,.\,(\ln x) - \ln y \,.\, \ln 3 = (\ln 3)^2 - (\ln 2)^2$ .... (i)

and $\ln x \,.\, \ln 3 - \ln y \,.\, \ln 2 = 0$ .... (ii)

Using (i) × $\ln 2$ – (ii) × $\ln 3$, we get

$\ln x \,.\, \{(\ln 2)^2 - (\ln 3)^2\} = (\ln 2)\{(\ln 3)^2 - (\ln 2)^2\}$

$\Rightarrow$ $\ln x = -\ln 2$ $\Rightarrow$ $x = 1/2$.

52. The value of $\displaystyle\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)}\,dx$ is

(A) (1 / 4) $\ln$ (3 / 2) (B) (1 / 2) $\ln$ (3 / 2) (C) $\ln$ (3 / 2) (D) (1 / 6) $\ln$ (3 / 2)

52. **(A)** $$I = \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)}\,dx$$

$x^2 = t$ $\Rightarrow$ $2x\,dx = dt$

$$\Rightarrow \quad I = \frac{1}{2}\int_{\ln 2}^{\ln 3} \frac{\sin t}{\sin t + \sin(\ln 6 - t)}\,dt$$

Using $\displaystyle\int_a^b f(x)dx = \frac{1}{2}\int_a^b (f(x) + f(a+b-x))dx$

$$\Rightarrow \quad I = \frac{1}{4}\int_{\ln 2}^{\ln 3}\left(\frac{\sin t}{\sin t + \sin(\ln 6 - t)} + \frac{\sin(\ln 6 - t)}{\sin(\ln 6 - t) + \sin t}\right)dt \quad [\text{since } \ln 3 + \ln 2 = \ln 6]$$

$$\Rightarrow \quad I = \frac{1}{4}\int_{\ln 2}^{\ln 3} dt = \frac{1}{4}[t]_{\ln 2}^{\ln 3} = \frac{1}{4}\ln\frac{3}{2}.$$

53. Let $\vec{a} = \quad + \hat{j} + \hat{k}$, $\vec{b} = \quad - \hat{j} + \hat{k}$, and $\vec{c} = \quad - \hat{j} - \hat{k}$ be three vectors. A vector $\overline{v}$ in the plane of and , whose projection on is $1/\sqrt{3}$ is given by

(A) $\quad - 3\hat{j} + 3\hat{k}$ (B) $-3\hat{i} - 3\hat{j} - \hat{k}$ (C) $3\hat{i} - \hat{j} + 3\hat{k}$ (D) $\hat{i} + 3\hat{j} - 3\hat{k}$

53. **(C)** $\overline{v} = \lambda(\quad + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

$\frac{\vec{v}.\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad (\lambda + \mu) - (\lambda - \mu) - (\lambda + \mu) = 1 \quad \Rightarrow \quad \mu = \lambda + 1$

$\overline{v} = \quad (2\lambda + 1) - \hat{j} + \hat{k}(2\lambda + 1)$

Only option (C) satisfies this one.

SECTION - II (Total Marks : 16)

(Multiple Correct Choice Type)

This section contains 4 multiple choice questions. Each question has four choices (A), (B), (C) and (D) out of which ONE OR MORE may be correct.

54. Let $f: R \to R$ be a function such that $f(x + y) = f(x) + f(y), \forall x, y \in R$.
If $f(x)$ is differentiable at $x = 0$, then
(A) $f(x)$ is differentiable only in a finite interval containing zero
(B) $f(x)$ is continuous $\forall x \in R$
(C) $f'(x)$ is constant $\forall x \in R$
(D) $f(x)$ is differentiable except at finitely many points

54. **(B)(C)** $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x)+f(h)-f(x)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$

Put $x = y = 0$, $f(0) = 0$
$f'(x) = f'(0)$
$f(x) = f'(0)\, x + c$
$f(x) = kx + c$, $c = 0$ as $f(0) = 0$
$\therefore$ Function is continuous. $f'(x)$ is constant.

55. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$.

If the hyperbola passes through a focus of the ellipse, then

(A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(B) a focus of the hyperbola is (2, 0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) the equation of the hyperbola is $x^2 - 3y^2 = 3$

55. **(B)(D)** $\frac{x^2}{4} + \frac{y^2}{1} = 1$, $a^2 = 4$, $b^2 = 1$, $e^2 = 1 - (b^2 / a^2) \Rightarrow e = \sqrt{3} / 2$

eccentricity of hyperbola = $2 / \sqrt{3}$.
Focus of ellipse $\equiv (\pm ae, 0) = (\pm \sqrt{3}, 0)$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through focus $(\pm \sqrt{3}, 0)$

$\therefore$ $a^2 = 3$ $\therefore$ $b^2 = a^2 (e^2 - 1) = 1$

$\therefore$ Equation of hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{3} - \frac{y^2}{1} = 1$

$x^2 - 3y^2 = 3$
Focus of hyperbola $\equiv (\pm ae, 0) = (\pm 2, 0)$.

56. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2 (M^TN)^{-1} (MN^{-1})^T$ is equal to

(A) M^2 (B) $-N^2$ (C) $-M^2$ (D) MN

56. **(C)** Since $M^T = -M$, $N^T = -N$

$M^2N^2 (M^TN)^{-1} (MN^{-1})^T$

$= M^2N^2 (-MN)^{-1} (MN^{-1})^T = M^2N^2 (N^{-1} . (-M)^{-1}) . (N^T)^{-1} . (-M)$

$= M^2N (-M)^{-1} . (-N)^{-1} . (-M) = M (MN) (MN)^{-1} (-M) = -M^2$.

57. The vector(s) which is/are coplanar with vectors $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$, and perpendicular to the vector $\hat{i} + \hat{j} + \hat{k}$ is / are

(A) $\hat{j} - \hat{k}$ (B) $-\hat{i} + \hat{j}$ (C) $\hat{i} - \hat{j}$ (D) $-\hat{j} + \hat{k}$

57. **(A)(D)** Let $\vec{a} = \lambda(\quad + \hat{j} + 2\hat{k}) + \mu(\hat{i} + 2\hat{j} + \hat{k})$

$= (\lambda + \mu)\hat{i} + (\lambda + 2\mu)\hat{j} + (2\lambda + \mu)\hat{k}$

$\vec{a} \cdot (\quad + \hat{j} + \hat{k}) = 0 \quad \therefore \quad \lambda = -\mu \quad \therefore \quad \vec{a} = \lambda(-\quad + \hat{k})$

Taking $\lambda = 1$, $\vec{a} = -\quad + \hat{k}$

$\lambda = -1$, $\vec{a} = \quad - \hat{k}$.

$\hat{i}\hat{j}$

SECTION - III (Total Marks : 15)

(Paragarph Type)

This Section contains **2 paragraphs**. Based upon one of the paragraphs **2 multiple choice questions** and based on other paragraph **3 multiple choice questions** have to be answered. Each of these has four choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

Paragraph for question Nos 58 and 59

Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls, and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now 1 ball is drawn at random from U_2.

58. The probability of the drawn ball from U_2 being white is
(A) 13 / 30 (B) 23 / 30 (C) 19 / 30 (D) 11 / 30

58. **(B)** $$P = \underbrace{\frac{1}{2}\left[\frac{3}{5}\times 1+\frac{2}{5}\times\frac{1}{2}\right]}_{\text{For appearing head}} + \underbrace{\frac{1}{2}\left[\frac{{}^3C_1\times{}^2C_1}{{}^5C_2}\times\frac{2}{3}+\frac{{}^3C_2}{{}^5C_2}\times 1+\frac{{}^2C_2}{{}^5C_2}\times\frac{1}{3}\right]}_{\text{For appearing tail}} = 23/30.$$

59. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is
(A) 17 / 23 (B) 11 / 23 (C) 15 / 23 (D) 12 / 23

59. **(D)** Using Bayes' theorem,

$$P = \frac{\frac{1}{2}\left[\frac{3}{5}\times 1+\frac{2}{5}\times\frac{1}{2}\right]}{\frac{1}{2}\left[\frac{3}{5}\times 1+\frac{2}{5}\times\frac{1}{2}\right]+\frac{1}{2}\left[\frac{{}^3C_1\times{}^2C_1}{{}^5C_2}\times\frac{2}{3}+\frac{{}^3C_2}{{}^5C_2}\times 1+\frac{{}^2C_2}{{}^5C_2}\times\frac{1}{3}\right]} = 12/23.$$

Paragraph for question Nos 60 and 62

Let a, b and c be three real numbers satisfying

$$[a \quad b \quad c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0 \quad 0 \quad 0] \qquad \text{...............} (E)$$

60. If the point $P(a, b, c)$, with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

(A) 0 (B) 12 (C) 7 (D) 6

60. **(D)** $D = \begin{vmatrix} 1 & 8 & 7 \\ 9 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0$

$\therefore$ system has non trivial solution.

If $c = k$, $b = -6/7\,k$, $a = -k/7$

$\because$ $2a + b + c = 1$

$-(2k/7) - (6k/7) + k = 1 \Rightarrow k = -7$

$\therefore$ P $(a, b, c) \equiv (1, 6, -7)$

$7a + b + c = 7 + 6 - 7 = 6.$

61. Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If $a = 2$ with b and c satisfying (E), then the value of $\dfrac{3}{\omega^a} + \dfrac{1}{\omega^b} + \dfrac{3}{\omega^c}$ is equal to

(A) -2 (B) 2 (C) 3 (D) -3

61. **(A)** $a = 2$, $k = -14$

$c = -14$, $b = 12$, $a = 2$

$$\frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}} = \frac{3}{\omega^2} + \frac{1}{(\omega^3)^4} + \frac{3(\omega^3)^5}{\omega}$$

$$= \frac{3}{\omega^2} + 1 + \frac{3}{\omega} = \frac{3 + \omega^2 + 3\omega}{\omega^2} = \frac{3(1+\omega)+\omega^2}{\omega^2} = \frac{-3\omega^2 + \omega^2}{\omega^2} = -2$$

62. Let $b = 6$, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)^n$ is

(A) 6 (B) 7 (C) 6 / 7 (D) ∞

62. **(B)** $b = 6$, $k = -7$, $c = -7$, $a = 1$

Q.E. becomes $x^2 + 6x - 7 = 0$; $a + b = -6$, $ab = -7$

$$\sum_{n=0}^{\infty}\left(\frac{\alpha+\beta}{\alpha\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n = 1 + \left(\frac{6}{7}\right) + \left(\frac{6}{7}\right)^2 + \ldots\ldots = \frac{1}{1 - \frac{6}{7}} = 7$$

SECTION - IV (Total Marks : 28)
(Integer Answer Type)

This section contains **7 questions**. The answer to each of the questions is a **single-digit integer**, ranging from 0 to 9. The bubble corresponding to the correct answer is to be darkened in the ORS.

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63. Let $f : [1, \infty) \to [(2, \infty)$ be a differentiable function such that $f(1) = 2$, if

$$6\int_1^x f(t)\,dt = 3x\,f(x) - x^3$$

for all $x \geq 1$, then the value of $f(2)$ is

63. **6.** Differentiating the given relation,

$6\,f(x) = 3\,f(x) + 3x\,f'(x) - 3x^2 \Rightarrow f'(x) - \dfrac{f(x)}{x} = x$, which is in linear form with I.F = $\dfrac{1}{x}$

Hence, $f(x) = x^2 + cx$ with $c = 1$. $\therefore f(x) = x^2 + x$. $\therefore f(2) = 6$.

64. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is

64. **5.** Expression $2\left|Z - 3 + \dfrac{5}{2}i\right|$ which is equivalent to double the distance between complex number z and $(3, -5/2)$ which lies on one the diameter $x = 3$ of circle given. Hence minimum value of expression = $2 \times \dfrac{5}{2} = 5$.

65. Let $a_1, a_2, \ldots a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i, 1 \leq p \leq 100$.

For any integer n with $1 \leq n \leq 20$, let $m = 5n$. If $\dfrac{S_m}{S_n}$ does not depend on n, then a_2 is

65. **9.** $$\frac{S_m}{S_n} = \frac{a_1 + a_2 + \ldots + a_{5n}}{a_1 + a_2 + \ldots + a_n} = \frac{\frac{5n}{2}\{2.3 + (5n-1)\,d\}}{\frac{n}{2}\{2.3 + (n-1)\,d\}} \quad ;$$

$$\lambda = \frac{5(6 + 5nd - d)}{(6 + nd - d)} \quad \left[\lambda = \frac{S_m}{S_n}\right]$$

$\therefore \lambda$ independent of n when $d = 6$

$\therefore a_2 = a_1 + d = 9$.

66. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is

66. **2.** Extremities of L.R. are $(2, 4)$ & $(2, -4) \Rightarrow \Delta_1 = 6$
Points of intersection of tangents at these points are $(-2, 0)$, $(1, 3)$ & $(-1, -1) \Rightarrow \Delta_2 = 3$.

67. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is.

67. **1.** Put $k = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)$ $\Rightarrow \tan k = \frac{\sin\theta}{\sqrt{\cos 2\theta}}$

$\Rightarrow \tan^2 k = \frac{\sin^2\theta}{1 - 2\sin^2\theta} \Rightarrow \sec^2 k - 1 = \frac{1}{\operatorname{cosec}^2\theta - 2}$

$\Rightarrow \sec^2 k = \frac{\operatorname{cosec}^2\theta - 1}{\operatorname{cosec}^2\theta - 2} \Rightarrow \sin^2 k = \frac{1}{\cot^2\theta} = \tan^2\theta$

$\Rightarrow \sin k = \tan\theta$ $\quad \therefore f(\theta) = \sin k = \tan\theta$. Hence. $\frac{d(f(\theta))}{d(\tan\theta)} = 1$

$\frac{2\sin x.\cos 2x}{\sin x.\sin 3x} = \frac{1}{\sin 2x}$

68. The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} with $a > 0$ is

68. **8**. Using A.M. $\geq$ G.M.,

$$\frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}}{8} \geq 1$$

$\therefore$ Minimum value of required expression = 8 (for $a = 1$).

69. The positive integer value of $n > 3$ satisfying the equation $\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is

69. **7**. $\frac{1}{\sin x} = \frac{1}{\sin 2x} + \frac{1}{\sin 3x}$ {where $x = (\pi / n)$}

or $\frac{\sin 3x - \sin x}{\sin x.\sin 3x} = \frac{1}{\sin 2x}$ or

$\sin 4x = \sin 3x$ [$\because \sin x \neq 0$]

$\therefore$ $4x = 3x$ (not possible, $\because$ $x \neq 0$)

or $4x = \pi - 3x$ $\Rightarrow$ $7x = \pi$ $\Rightarrow$ $7.(\pi / n) = \pi$ $\therefore n = 7$.