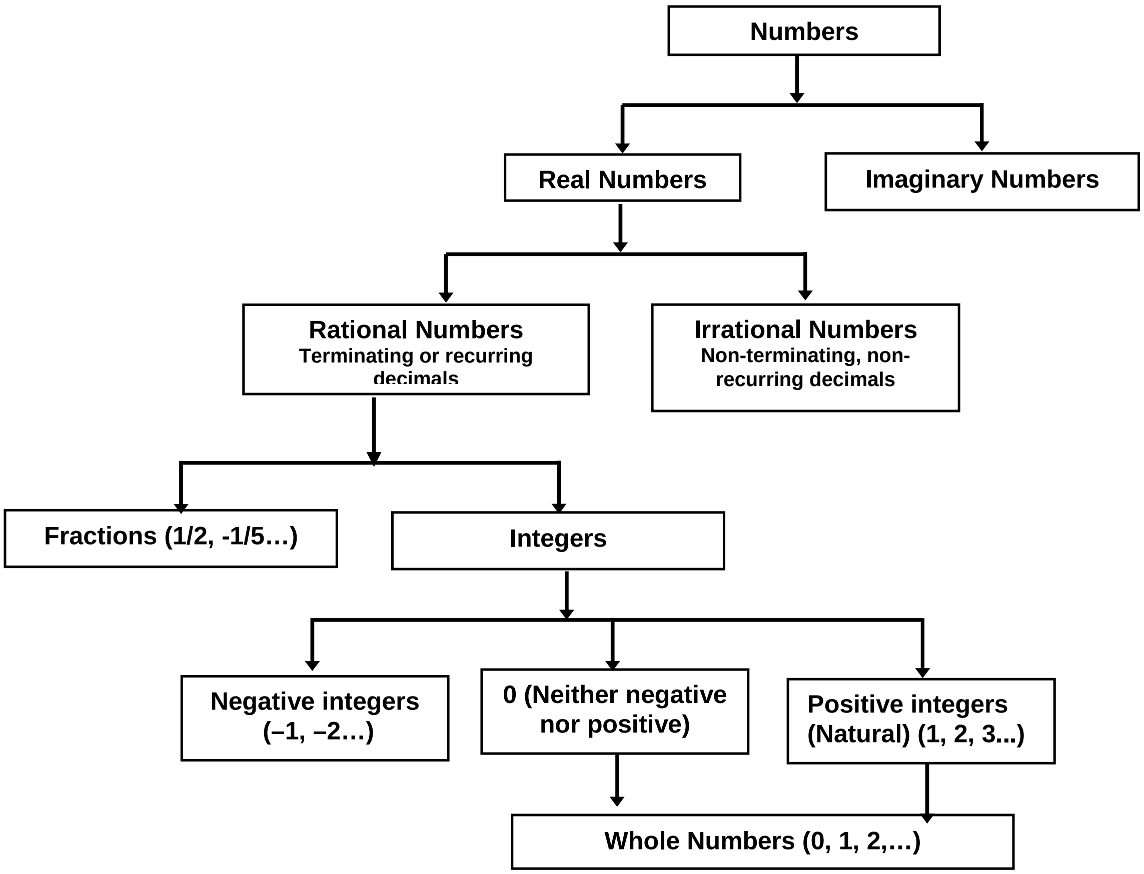
Number Theory

Number system



Prime numbers: Numbers which have exactly 2 factors (1 and number itself): Eg: 2, 3, 5, 7, 11,

Composite numbers: Numbers which have more than 2 factors

Remember: i. 1 is neither prime nor composite.

ii. If a number 'N' is not divisible by any prime number less than $\sqrt{N}$, then N is a prime number.

iii. Every prime number greater than 3 can be written in the form of (6k + 1) or (6k – 1), where k is an integer.

Relative primes: Numbers which do not have common factor other than 1. Eg: 3 and 8, 15 and 16.

Perfect numbers: If the sum of all the factors excluding itself (but including 1) is equal to the number itself, then the number is called perfect number. E.g. 6, 28

Note : .i. The product of 2 consecutive integers is always divisible by 2.

ii. The product of n consecutive integers is always divisible by n!

Pure recurring decimal: if all the digits after decimal repeat, then it is called pure recurring.

<u>**Converting pure recurring decimal to fraction**</u>

Ex: 0.abababab…. = $\frac{ab}{99}$ **. i.e.,** $\frac{\text{recurring digits}}{\text{as many 9's as the number of recurring digits}}$

Converting mixed recurring decimal to fraction

Ex: **0.abcbcbcbc… =** $\frac{bc-a}{990}$ **i.e.,**

$$\frac{\text{recurring digits} - \text{non recurring digits}}{\text{as many 9's as the no. of recurring digits followed by as many 0's as the no. of non-recurring digits}}$$

Divisibility rules

Test for divisibility by 2	The last digit should be divisible by 2.
Test for divisibility by 3	The sum of digits should be divisible by 3
Test for divisibility by 4	The number formed with its last 2 digits should be divisible by 4
Test for divisibility by 5	The last number should be divisible by 5
Test for divisibility by 6	It should be divisible by both 2 and 3.
Test for divisibility by 7	Subtract two times the unit digit from the remaining number. If it is divisible by 7, then the number is also divisible by 7.
Test for divisibility by 8	The number formed by its last 3 digits should be divisible by 8.
Test for divisibility by 9	The sum of the digits should be divisible by 9.
Test for divisibility by 10	The last digit should be 0.
Test for divisibility by 11	Subtract the unit digit from the remaining number.
Test for divisibility by 13	Add four times the unit digit to the remaining number
Test for divisibility by 17	Subtract five times the unit digit from the remaining number.
Test for divisibility by 19	Add two times the unit digit to the remaining number.
Test for divisibility by 23	Add seven times the unit digit to the remaining number.

Number/Sum of factors:

If a number N is written as N = $a^p \times b^q \times c^r \times \ldots$, where a, b, c are prime numbers, then

- The number of factors of 'N' is (p + 1) (q + 1)
- Similarly, the sum of factors of 'N' = $\left(\frac{a^{p+1}-1}{a-1}\right)\left(\frac{b^{q+1}-1}{b-1}\right)$ ……
- The number of ways of writing the given number as a product of 2 factors= $\frac{1}{2}(p+1)(q+1)$….
- If N is a perfect square, 2 cases will come.

Case 1: Number of ways of writing N as a product of 2 different factors= $\frac{1}{2}$ {(p+1)

Case 2: Number of ways of writing N as a product of 2 factors= $\frac{1}{2}\{(p+1)(q+1).....+1\}$

- Number of co –primes to N which are less than N = $N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)$ …..

- Number of ways of writing N as a product of 2 co-primes= 2^{n-1}, where, n is the number of different prime factors to N.

- Sum of all the numbers, co-primes to N, which are less than N = $\frac{N}{2}\ N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)$ …..

Find the remainders using Binomial and Congruent Modulo

(e.g Find the remainder when 7^{25} divided by 6)

Binomial Theorem:

$(x + y)^n = {}^nc_0\, x^n + {}^nc_1 x^{n-1}.y + {}^nc_2 x^{n-2}.y^2 + ….. + {}^nc_n y^n$. Where ${}^nc_0, {}^nc_1 ….. {}^nc_n$ are binomial coefficients.

${}^nc_r = \frac{n!}{r!(n-r)!}$.

Congruent Modulo

a is said to be congruent to b , if they leave same remainder when divided by n

a = b(mod n) means a – b is a multiple of n.

Ex: 26 = 4(mod 11), because 26 – 4 = 22 is divisible by 11.

Note: If $a_1 = b_1$ (mod n) and $a_2 = b_2$ (mod n) then

$a_1 + a_2 = (b_1+b_2)$ (mod n)

$a_1 – a_2 = (b_1-b_2)$ (mod n)

$a_1 \times a_2 = (b_1 \times b_2)$ (mod n)

Ex. What is the remainder, when 2^{256} is divided by 17?

(1) 1 (2) 16 (3) 14 (4) 10 (5) None of these

Sol. We can write 17 as $2^4 + 1$ and 2^{256} as $(2^4)^{64}$.

[If f(x) is divided by (x – a), the remainder is f(a)]

$\therefore$ The remainder is $(-1)^{64} = 1$. **Answer: (1)**

Ex. What is the remainder, when 15^{75} is divided by 7?

(1) 1 (2) 2 (3) 6 (4) 0 (5) None of these

Sol. When 15 is divided by 7, the remainder is 1.

So, the answer is $1^{75} = 1$. **Answer: (1)**

Finding the Highest power of the number contained in the factorial of given number

e.g Find the largest power of 3 that can divide 95! Or Finding the number of Zeros in n!

Approach: Find the largest power of 3 contained in 95!.

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3 | 95
3 | 31 ---> Quotient
3 | 10 ---> Quotient
3 | 3  ---> Quotient
  | 1  ---> Quotient
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Add all the quotients 31 + 10 + 3 + 1, which give 45.

Remainder for the numbers of the form $a^n + b^n$ or $a^n - b^n$.

	If n is even	If n is odd
$a^n - b^n$	divisible by (a – b) and (a + b)	divisible by (a – b)
$a^n + b^n$		divisible by (a + b)

Try to solve these questions by using above results:

1. Let N = $55^3 + 17^3 - 72^3$. N is divisible by

   (1) both 7 and 13 (2) both 3 and 13

   (3) both 17 and 7 (4) both 3 and 17

   (5) none of these

2. If x = $(16^3 + 17^3 + 18^3 + 19^3)$, what is the remainder when x is divided by 70?

   (1) 0 (2) 1 (3) 69 (4) 35 (5) None of these

Finding last digit or unit digit in a^b

Remember: Last digit of a product of numbers = the product of last digits

Step 1: Divide b (only last two digits if number of digits more than 3) by 4, check the remainder

Step 2 : If remainder is 0, then the unit digit is last digit of $(\text{unit digit of a})^4$

If remainder is 1, then the unit digit is last digit of $(\text{unit digit of a})^1$

If remainder is 2, then the unit digit is last digit of $(\text{unit digit of a})^2$

If remainder is 3, then the unit digit is last digit of $(\text{unit digit of a})^3$

Remember: if unit digit of 'a' is 5 or 6, the last digit is always 5 or 6, respectively.

For 4 and 9, if the power is odd, the last digits are 4 and 9, respectively and if the power is even, the last digits will be 6 and 1, respectively.

Solve this : What is the right most non-zero digit of $(30)^{2740}$?

(1) 1 (2) 3 (3) 7 (4) 9 (5) None of these

Answer: (1)

Base conversions:

Converting from other number bases to decimal

The value of the number 12304 in base ‘a’ is determined by computing the place value of each of the digits of the number:

	1	2	3	0	4	number
Add these →	a^4	a^3	a^2	a^1	a^0	place values

Converting from decimal to other number bases

One way to do this is to repeatedly divide the decimal number by the base in which it is to be converted, until the quotient becomes zero. As the number is divided, the remainders - in reverse order - form the digits of the number in the other base.

Example: Convert the decimal number **82** to base 6:

82/6 = 13 remainder 4

13/6 = 2 remainder 1

2/6 = 0 remainder 2

The answer is formed by taking the remainders in reverse order: 2 1 4 base 6

Least Common Multiple(LCM) and Highest Common Factor(HCF)

HCF divides the numbers and numbers divides the LCM

L.C.M H.C.F = product of the two numbers.

L.C.M is always multiple of H.C.F.

LCM of fractions = $\frac{\text{LCM of numerators}}{\text{HCF of deno min ators}}$

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Question	Approach
Find the least number, which is exactly divisible by x, y, z.	LCM (x, y, z)
Find the least number, which when divided by x, y, z, leaves a remainder ‘r’ in each case.	LCM (x, y, z) + r
Find the least number, which when divided by x, y, z leaves remainders a, b, c respectively.	Observe, if x – a = y – b = z – c = k (say). Then LCM (x, y, z) – k

	Else, go with the options.
Find the greatest number, that will exactly divide x, y, z.	HCF (x, y, z)
Find the greatest number, that will divide x, y, z leaving remainders a, b, c respectively.	HCF (x – a, y – b, z – c)
Find the greatest number, that will divide x, y, z leaving the same remainder in each case.	HCF (x – y, y – z, z – x)

Examples

Ex. A red light flashes 3 times per minute and a green light flashes 5 times in two minutes at regular intervals. If both lights start flashing at the same time, how many times do they flash together in each hour?

(1) 30 (2) 24 (3) 20 (4) 60 (5) 90

Hint: LCM of (20, 24) = duration after which both the light will flash together.

Ex. Three wheels can complete 60, 36, 24 revolutions per minute respectively. There is red spot on each wheel that touches the ground at time zero. After how much time, all these spots will simultaneously touch the ground again?

(1) $\frac{5}{2}$ sec (2) $\frac{5}{3}$ sec (3) 5 sec (4) 7.5 sec (5) 9 sec

Sol. Find the time taken by the three wheels to complete one revolution and take their LCM

Answer: (3)

Ex. A is the set of positive integers such that, when divided by 2, 3, 4, 5, 6 leaves the remainders 1, 2, 3, 4, 5 respectively. How many integers between 0 and 100 belong to set A?

(1) 0 (2) 1 (3) 2 (4) 3 (5) None of these

Sol. The least such number is LCM (2, 3, 4, 5, 6) – 1

= 60 – 1 = 59

The next number is 59 + 60 = 119.

So, only one number lies below 100. **Answer: (2)**