ALGEBRA

Quadratic Equations

Remainder theorem: If f(x) is divided by(x – a), the remainder is f(a).

Factor theorem: If (x – a) is a factor of f(x), then f(a) = 0.

Involution and Evalution

- $(a + b)^2 = a^2 + b^2 + 2ab$
- $(a - b)^2 = a^2 + b^2 - 2ab$
- $(a + b)^3 = a^3 + b^3 + 3ab\,(a + b)$
- $(a - b)^3 = a^3 - b^3 - 3ab\,(a - b)$
- $a^2 - b^2 = (a - b)\,(a + b)$
- $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
- $a^3 + b^3 = (a + b)\,(a^2 - ab + b^2)$
- $a^3 - b^3 = (a - b)\,(a^2 + ab + b^2)$
- $a^3 + b^3 + c^3 - 3abc = (a + b + c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$
- if $a + b + c = 0$
- $a^3 + b^3 + c^3 = 3abc$

Linear Equations

A pair of linear equations in two variables, say x and y, is said to form a **system of simultaneous linear equations in two variables.**

The general form of a system of linear equations in two variables x and y is

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

Set of Equations

Consistent System means have solution

Inconsistent System means have no solution $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Dependent system has infinitely many solutions

Condition: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Independent system: Only one unique solution

Condition: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Quadratic Equations

- General form of quadratic equation is

$ax^2 + bx + c = 0$

Roots are $\frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$

Sum of roots = $\frac{-b}{a}$

Product of roots = $\frac{c}{a}$

Nature of roots

- If $b^2 - 4ac = 0$ real and equal
- $b^2 - 4ac > 0$ real and distinct
- $b^2 - 4ac < 0$ imaginary

Forming Equation from roots:

If α and β are the roots of any quadratic equation then that equation can be written in the form

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

i.e. x^2 – (sum of the roots) x + Product of the roots = 0.

Cubic Equations

Cubic equation $ax^3 + bx^2 + cx + d = 0$ will have three roots (say x_1, x_2, x_3). Then

- Sum = $x_1 + x_2 + x_3 = \frac{-b}{a}$
- Sum of Products (taken 2 at a time) = $x_1x_2 + x_2x_3 + x_3x_1 = \frac{c}{a}$
- Product = $x_1\, x_2\, x_3 = \frac{-d}{a}$

<u>Important:</u> If the graph of given equation cuts x-axis n times then it will have n real roots.

Ex.1 What is the value of $\sqrt{n + \sqrt{n + \sqrt{n + \ldots\ldots\ldots}}}$

Sol. $x = \sqrt{n + x}$

$\Rightarrow x^2 = n + x$

$x^2 - x - n = 0$

$x = \frac{1 + \sqrt{1 + 4n}}{2}$

Ex.2 $y = \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\ldots\ldots}}}}$. What is the value of y?

(1) $\frac{\sqrt{13}+3}{2}$ (2) $\frac{\sqrt{13}-3}{2}$ (3) $\frac{\sqrt{15}+3}{2}$ (4) $\frac{\sqrt{15}-3}{2}$ (5) None of these

Answer: (4)

Ex.3 One root of $x^2 + kx - 8 = 0$ is square of the other, then, the value of k is

(1) 2 (2) 8 (3) – 8 (4) – 2 (5) 0

Hint: Let the roots be α, α^2.

Inequalities, Maxima and Minima

Intervals

(a, b) means $a < x < b$

(a, b] means $a < x \le b$

[a, b] means $a \le x \le b$

- **For inequalities questions always go with the options and eliminate the wrong options by taking values for the variable in the given range.**

Some Important Points

- If $a > b$ and $b > c$, then $a > c$.
- If $a > b$ and c is any real number, then $a + c > b + c$ and $a - c > b - c$.
- If $a > b > 0$, then $\frac{1}{a} < \frac{1}{b}$.
- If $x > y > 0$, then $\log_a x > \log_a y$, if $a > 1$ and $\log_a x < \log_a y$, if $0 < a < 1$.
- A.M $\ge$ G.M $\ge$ H.M
- $a^2 + b^2 + c^2 \ge ab + bc + ca$
- $(n!)^2 > n^n$ for $n > 2$.
- $2 \le \left(1+\frac{1}{n}\right)^n \le 3$ for any integer n.
- $a^2b + b^2c + c^2a \ge 3abc$.
- $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \ge 4$
- $a^4 + b^4 + c^4 + d^4 \ge 4abcd$.
- If $(x - a)(x - b) \ge 0$, $a < b$, then $x \le a$ or $x \ge b$.

- If $(x-a)(x-b) \le 0$, a< b, then $a \le x \le b$.
- If $|x-a| > b$ then $x > a + b$ or $x < a - b$
- If $|x-a| < b$, then $a - b < x < a + b$.
- $|a + b| \le |a| + |b|$
- $|a - b| \ge |a| - |b|$

Ex.4 Largest value of min $(2 + x^2, 6 - 3x)$, when $x > 0$, is

Sol. For this type of questions,

Take, $2 + x^2, 6 - 3x$

$\Rightarrow x^2 + 3x - 4 = 0$

$\therefore x = -4, 1$. (Since it is given $x > 0 \Rightarrow x = 1$)

Put $x = 1$ in either $2 + x^2$ or $6 - 3x$

So, the answer is $2 + 1^2 = 3$.

Maximum /Minimum value of Quadratic Equation

1. If the sum of two quantities is constant, then the product will be maximum, if both are equal.
2. If the product is constant, then the sum will be minimum, if both are equal.
3. The equation $ax^2 + bx + c = 0$, will have maximum value when $a < 0$ and minimum value when $a > 0$.

   The maximum or minimum values are given by $\frac{4ac - b^2}{4a}$, and will occur at $x = \frac{-b}{2a}$.

Functions

- If $f(x) = x^3 - 3x^2 + 2x$, then $f(a) = a^3 - 3a^2 + 2a$.
- $fog(x) = f\{g(x)\}$
- $gof(x) = g\{f(x)\}$
- If $f(x) = f(-x)$, then f(x) is an even function. E.g. x^2
- If $f(x) = -f(-x)$, then f(x) is an odd function. E.g. x^3
- If $f(x) = y \Rightarrow x = f^{-1}(y)$

Finding the inverse of a given function.

Ex.5 ***Find the inverse of*** $y = \frac{-2}{x-5}$ ***and determine whether the inverse is also a function.***

Sol. Since the variable is in the denominator, this is a rational function. Here's the algebra:

Step 1: Write the original function $y = \frac{-2}{x-5}$

Step 2: Represent x in terms of y in the equation. $x = \frac{5y-2}{y}$

Step 3: Replace the x's and y's with each other. $y = \frac{5x-2}{x}$

Thus, the inverse function is $y = \frac{5x-2}{x}$.

Important: If f and g are two functions defined from set A in to set B, then

1. Sum / difference of two functions is **$(f \pm g)(x) = f(x) \pm g(x)$**
2. Product of two functions is **$(f \times g)(x) = f(x) \times g(x)$**
3. Division of two functions is $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

Points on graph

- If the graph is symmetrical about y-axis then it is even function.
- If the graph is symmetrical about origin then it is odd function.
- **For graph questions it is always better to take values and check the options.**

Some Important graphs

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Modulus Function

$f(x) = |x|$ or $f(x) = -x$ when $x < 0$

x when $x \geq 0$

Domain: R

Range: R^+

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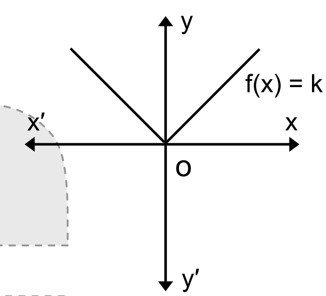

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Greatest Integer Function or Step Function

$f(x) = [x]$

Domain: R

Range: Integer

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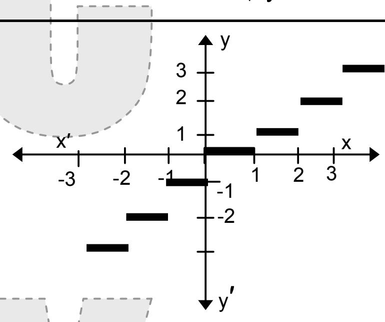

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Reciprocal function

$f(x) = \frac{1}{x}$

Domain: $R - \{0\}$

Range: $R - \{0\}$

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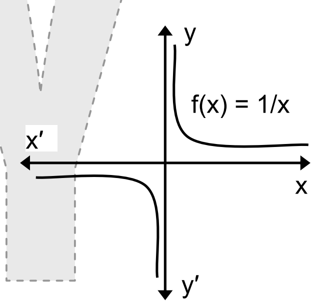

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Logarithms

Properties of Logarithms

- $\log_1 1 = 0$
- $\log_a a = 1$
- $\log_a a^m = m$
- $\log_{a^n} a = \frac{1}{n}$
- $\log_{b^n} a^m = \frac{m}{n} \log_b a$
- $\log_a m + \log_a n = \log_a mn$
- $\log_a m - \log_a n = \log_a m/n$
- $\log_b a = \frac{\log a}{\log b} = \frac{1}{\log_a b}$
- $a^{\log_a m} = m$
- If $\log_a N = x$, then $N = a^x$.