ACTUARIAL SOCIETY OF INDIA

November 2005 Examinations

SUBJECT CT-6 : STATISTICAL MODELS **Total Marks 100**

Solution 1.

Short-tailed business refers to lines of general insurance business in which the claims are settled quickly. Long-tailed business refers to lines of general insurance business in which claims generally take a long time to be settled.

[1]

Example:
Short-tailed business: Property damage
Long-tailed: Liability business.

[1]
[Total 2]

Solution 2.

It is easy to see that ln $X_t = a + bt + z_t$. Therefore,

$$\begin{aligned} \ln X_t \quad \ln X_{t-1} &= (a + bt + Z_t) \quad [a + b(t \quad 1) + Z_{t-1}] \\ &= b + (Z_t \quad Z_{t-1}). \end{aligned}$$

Since $Z_t \quad Z_{t-1}$ is $I(0)$, the time series $Y_1, Y_2, \ldots$, defined by

$$Y_t = \ln X_t \quad \ln X_{t-1},$$

is stationary.

[Total 2]

Solution 3.

(i) The surplus at time t is

$$U(t) = u + (1 + q)\, l m t - \sum_{i=1}^{N(t)} X_i$$

[1]

(ii) The probability of ruin is:

$$y(u) = P(U(t) < 0) \quad for \quad some \quad t, 0 < t < \infty$$

When $q = 0 \quad y(u) = 1, \quad u > 0$

[1]

(iii) The colleague's statement is not correct. As the value of λ increases the rate of claims arrival will increase. This will be accompanied by a correspondingly increased rate of accumulation of premium – effectively altering the time-scale of the overall cash flow. However this will not alter the size of each claim. So the time at which ruin occurs may be altered but not the probability of ultimate ruin.

[2]

(iv) The surplus process ignores:

- investment income on the cash flows.
- insurer's expenses.
- delays in claim settlement
- any reinsurance arrangement

[2]

[Total 6]

Solution 4.

(i) Here, $F(x) = 1 \quad exp(\text{-}10x^3)$. Therefore,
$F^{-1}(u) = [\quad 1/10 \ln(1 \quad u)]^{1/3}$

The transformed variates $F^{-1}(U_1)$, $F^{-1}(U_2)$, $F^{-1}(U_3)$, . . . have the requisite distribution.

[2]

(ii) Use acceptance-rejection method. Let $V_1 = ðU_1$, so that V_1 is uniformly distributed on [0, ð] and has density function $g(x) = 1/ð$ over that range. This is the 'dominating density'.

[1]

Define

$$C = \sup_{0<x<p} \frac{f(x)}{g(x)} = \sup_{0<x<p} 2\sin^2 x = 2$$

If $U_2 < \sin^2 V_1$ let $X_1 = V_1$; otherwise reject this value and select a new pair U_1, $U2$ and try again.
Repeat for other X_i.

[2]

(iii) Divide [0, 1] into equal-length intervals [0, $1/n$], ($1/n$, $2/n$], . . .,(1 $\quad 1/n$, 1]. Pick X_i if $(i \quad 1)/n < U_i \quad i/n$.

[2]

[Total 7]

Solution 5.

(i) The general form can be written as
$C_{ij} = r_j\ s_i\, x_{i+j} + e_{ij}$

where C_{ij} is incremental claims and
r_j is the development factor for year j, independent of origin year i

s_i is a parameter varying by origin year, representing exposure

x_{i+j} is a parameter varying by calendar year, representing inflation

e_{ij} is an error term

[3]

(ii) Using the incurred claims, we can calculate the cumulative incurred claims:

Cumulative incurred claims

Accident Year		***Delay Year***	
2002	4,253	5,208	5,443
2003	3,142	5,087	
2004	4,002		

[1]

The development factors are:

5,443/5,208 = 1.04512
and (5,087 + 5,208) / (3,142 + 4,253) = 1.39216

[1]

1 – 1/*f* = 1 – 1/(1.04512*1.39216) = 0.3127

[1]

2004 Emerging Liability is = 4,500 * 0.90 * 0.3127 = 1,266

[1]

Reported Liability = 4,002
Ultimate Liability = 5,268
Reserve = 5,268 – 1,885 = 3,383

[1]
[Total 8]

Solution 6.

(i) Smallest loss amount:

Discount	**If claim**	**If no claim**	**Difference**
0%	900, 675, 495	675, 495, 360	**540**

So the smallest loss for which claim will be made at the 0% level is 540.

[1]

(ii)

$$P(Claim) = P(Claim \mid Accident)P(Accident)$$

$$= P(X > x) * 0.2$$

where X is the loss, which has a lognormal distribution, and x is the minimum loss for which a claim will be made.

[1]

$$E(X) = \exp(m + 1/2s^2) = 1{,}188$$
$$V(X) = \exp\{2(m + 1/2s^2)\}.[\exp(s^2) - 1] = (495)^2$$

[1]

$$\therefore \exp(s^2) - 1 = \frac{495^2}{1188^2}$$

$$s^2 = 0.16$$

Hence, $s = 0.4, m = 7$

[1]

$$P(X > x) = 1 - \Phi\left(\frac{\ln(x) - m}{s}\right) = 1 - \Phi\left(\frac{\ln(x) - 7}{0.4}\right)$$

$$P(X > 540) = 1 - \Phi(-1.771) = \Phi(1.771) = 0.9617$$

$$P(Claim) = 0.91671 \times 0.2 = 0.1923$$

[1]

The transition matrix can now by completed:

$$\begin{pmatrix} 0.192 & 0.808 & 0 & 0 \\ 0.147 & 0 & 0.853 & 0 \\ 0.120 & 0 & 0 & 0.880 \\ 0 & 0.197 & 0 & 0.803 \end{pmatrix}$$

[1]

(iii) The steady state distribution is now the solution of:

$$0.192p_0 + 0.147p_1 + 0.120p_2 = p_0$$
$$0.808p_0 + 0.197p_3 = p_1$$
$$0.853p_1 = p_2$$
$$0.880p_2 + 0.803p_3 = p_3$$
$$p_0 + p_1 + p_2 + p_3 = 1$$

[1]

Expressing p_0, p_1, p_3 in terms of p_2:

$$p_3 = \frac{0.880}{0.197} p_2 = 4.4670 p_2$$

$$p_1 = \frac{1}{0.853} p_2 = 1.1723 p_2$$

$$p_0 = \frac{0.147/0.853 + 0.120}{0.808} p_2 = 0.3618 p_2$$

$$\therefore 0.3618p_2 + 1.1723p_2 + p_2 + 4.4670p_2 = 1$$

$$p_2 = 0.1428$$

[1]

The proportions at each level of discount in the steady state are therefore:

0%: 5.2%
25%: 16.7%
45%: 14.3%
60%: 63.8%
Total: 100.0%

[1]

[Total 9]

Solution 7.

Posterior density of *è* is

$$\frac{\frac{1}{q} I(X < q) q e^{-q}}{\int \frac{1}{q} I(X < q) q e^{-q}\, dq} = \frac{I(X < q) e^{-q}}{\int_X^{\infty} e^{-q}\, dq} = \begin{cases} e^{-(q-X)} & if q > X \\ 0 & otherwise \end{cases}$$

[2]

(i) Posterior mean of è is
$E(è|X) = X + E(è \quad X|X) = X + 1.$

Bayes estimator of *è* with respect to the squared error loss function is $X + 1$

[2]

(ii) Bayes estimate of *è* with respect to the absolute error loss function is the median of the posterior distribution. This is given by the solution to the equation

$$\frac{1}{2} = \int_X^x e^{-(q-X)}\, dq = 1 - e^{-(q-X)}$$

The solution is $X + ln\, 2$.

[2]

(iii) Likelihood of *è* is

$$\prod_{i=1}^{2} \frac{1}{q} I(q > X_i) = \frac{1}{q} I(q > Max\{X_1, X_2\})$$

[1]

The posterior density is

$$\frac{\frac{1}{q^2} I(q > Max\{X_1, X_2\}) q e^{-q}}{\int_{\max(X_1, X_2)}^{\infty} \frac{1}{q} e^{-q}\, dq}$$

[1]

Bayes estimator or the posterior mean is given by

$$\frac{\int_{\max(X_1,X_2)}^{\infty} e^{-q}\, dq}{\int_{\max(X_1,X_2)}^{\infty} \frac{1}{q} e^{-q}\, dq} = \frac{\exp(-Max(X_1, X_2))}{\int_{\max(X_1,X_2)}^{\infty} \frac{1}{q} e^{-q}\, dq}$$

$$= \left[\int_{\max(X_1,X_2)}^{\infty} \frac{1}{q} \exp\{-(q - \max(X_1, X_2))\} dq \right]^{-1}$$

which further simplifies to

$$= \left[\int_0^{\infty} \frac{e^{-y}}{(y + Max(X_1, X_2))} dy \right]^{-1}$$

[1]

[Total 9]

Solution 8.

(i) Mean of the gamma density is *áâ*, which must be a function of the canonical parameter *è*. The gamma density can be rewritten as

$$f(y) = y^{a-1} b^{-a} \exp(-y/b)/\Gamma(a)$$

$$= \exp[(a-1)\log y - a \log b - y/b - \log\Gamma(a)]$$

The term log *y* must be absorbed in *c*(*y*, *ö*), therefore we can choose *f* = *á*. Turning to *y/â*, we can rewrite it as *yá/*(*áâ*). Now it is clear that *è* and *a*(*ö*) should be 1/(*áâ*) and 1/*á*, respectively (or with interchanged signs). By rearranging the terms we have the requisite form of exponential family density with

è = 1/(*áâ*),

f = *á*,

b(*è*) = log(*è*),

a(*f*) = 1/ *f*,

c(*y*, *f*) = *á* log *á* + (*á* 1) log *y*.

[3]

(ii) *E*(*Y*) = b'(*è*) = 1/*è* = *áâ*.

[1]

$Var(Y) = a(f)b''(è) = (\ \ 1/á)(\ \ (áâ)^2) = áâ^2.$

[1]

(iii) The canonical link function is the reciprocal function (inverse of the b function). Therefore, the model for the claim size Y is

$$\frac{1}{E(Y)} = b_0 + \sum_{i=1}^{4} b_i x_i$$

This GLM can be fitted into any statistical package.

[2]

In terms of the parameters of this model, the hypothesis of “no gender effect on claim size” translates into $\hat{a}_2 = 0$. The t-statistic associated with $\hat{a}_2$ is the appropriate test statistic. The appropriate degrees of freedom is $n \quad 5$, where n is the number of observations.

[2]

[Total 9]

Solution 9

(i) $$f(x) = F'(x) = \frac{a\,100^a}{(100+x)^{a+1}}$$

[1]

The likelihood function is:

$$L(a) = \prod_{i=1}^{h} \frac{a\,100^a}{(100+x_i)^{a+1}} \times \left(\frac{100}{100+M}\right)^{a(n-h)}$$

$$= \frac{a^h 100^{an}}{(100+M)^{a(n-h)}} \prod_{i=1}^{h} (100+x_i)^{-a-1}$$

[1]

$$l(a) = \log L(a) = h\log a + an\log 100 - a(n-h)\log(100+M) - (a+1)\sum_{i=1}^{h}\log(100+x_i)$$

[1]

$$\frac{\partial l}{\partial a} = \frac{h}{a} + n\log(100) - (n-h)\log(100+M) - \sum_{i=1}^{h}\log(100+x_i)$$

$\dfrac{\partial^2 l}{\partial a^2} = -\dfrac{h}{a^2} < 0$ and is hence a maximum.

[1]

Equating $\dfrac{\partial l}{\partial a} = 0$ we obtain

$$\hat{a} = \frac{h}{(n-h)\log(100+M) - n\log(100) + y}$$

[1]

(ii)

We require $E(Y) = \int_0^M xf(x)dx + MP(X > M)$

[1]

$$= E(X) - \int_M^\infty (x-M)f(x)dx$$

$$E(X) = \frac{l}{a-1} = \frac{100}{0.5} = 200$$

[1]

$$\int_M^\infty (x-M)f(x)dx = \int_0^\infty zf(z+M)dz$$

$$= \frac{100^a}{600^a}\int_0^\infty \frac{za\,600^a}{(600+x)^{a+1}}dz$$

$$= \left(\frac{1}{6}\right)^a \frac{l'}{a-1} = \frac{600}{7.3485} = 81.65$$

[2]

Thus $\quad E(Y) = 200 - 81.65 = 118.35$

[1]

[Total 10]

Solution 10.

(i) Since $|á| < 1$, the process is ARMA(1,1).

[1]

Therefore, $p = q = 1$ and $d = 0$.

[1]

(ii) The time series is itself stationary, so no differencing is necessary. We have (1 $áB$)Y = [1+(1 $á$)B]Z, i.e.,

$$Y = \frac{1+(1-a)B}{1-aB}Z$$

$$= \left[1+(1-a)B\right].\left[1+aB+a^2B^2+a^3B^3\ldots\right]\ Z$$

$$= \left[1+B+aB^2+a^2B^3+a^3B^4+\ldots\right]\ Z.$$

[2]

It follows that

$V ar(Yi) = ã_0 = (1 + 1 + á^2 + á^4 + á^6 \cdot \cdot \cdot)ó^2$

$= [1+1/(1 \quad á^2)]ó^2;$

$Cov(Yi, Yi+k) = ã_k = (á^{k-1} + á^k + á^{k+2} + \cdot \cdot \cdot)ó^2$

$= [á^{k-1} + á^k/(1 \quad á^2)]\, ó^2$

[2]

Therefore,

$$ñk = ã_k/ ã_0 = \frac{a^{k-1}(1-a^2)+a^k}{(1-a^2)+1}. = \frac{a^{k-1}+a^k-a^{k+1}}{2-a^2}$$

[2]

(iii) It follows from part (ii) that $ñ_1 = (1+á \quad á^2)/(2 \quad á^2)$. Set this equal to the empirical version, r_1. This equation simplifies to the quadratic equation

$$(1 \quad r_1)\, á^2 \quad á + (2\, r_1 \quad 1) = 0.$$

The estimator of $á$ is a root of this quadratic equation.

[1]

Since $|r_1| < 1$, the coefficient of $á^2$ is positive. There is a solution to the quadratic equation if the discriminant, $1 \quad 4(1 \quad r_1)(2\, r_1 \quad 1)$, is positive. It is easy to see that the discriminant takes the minimum value of 1/2 when $r_1 = 3/4$, hence it is *always* positive.

The root $\dfrac{1+[1-4(1-r_1)(2r_1-1)]^{1/2}}{2(1-r_1)}$ is always greater than 1, while the other root

$\dfrac{1-[1-4(1-r_1)(2r_1-1)]^{1/2}}{2(1-r_1)}$ is always between –1 and +1. Hence, the latter root should be chosen.

[1]

[Total 10]

Solution 11.

(i) (a) $f(x) = 0.75\, f_A(x) + 0.25\, f_B(x)$

$$P(X > 2000) = 0.75\int_{2000}^{\infty} f_A(x)dx + 0.25\int_{2000}^{\infty} f_B(x)dx$$

$$= 0.75\left(\frac{200}{200+2000}\right)^{3.5} + 0.25\left(\frac{1200}{1200+2000}\right)^{4} = 0.00511$$

[3]

(b) $E(X) = 0.75 \times \dfrac{200}{2.5} + 0.25 \times \dfrac{1200}{3} = 160$

[1]

$$E(X^2) = 0.75 \times \left[\left(\frac{200}{2.5} \right)^2 + \frac{3.5 \times 200^2}{2.5^2 \times 1.5} \right] + 0.25 \times \left[\left(\frac{1200}{3} \right)^2 + \frac{4 \times 1200^2}{3^2 \times 2} \right]$$

$$= 136{,}000$$

$$V(X) = 136{,}000 - 160^2 = 110{,}400$$

[2]

(ii) $$\frac{l}{a-1} = 160 \quad and \quad \frac{a\,l^2}{(a-1)^2(a-2)} = 110{,}400$$

[1]

$$\therefore \frac{a}{a-2} = \frac{110{,}400}{160^2} = 4.3125$$

$$a = 2.6038 \quad and \quad l = 256.6038$$

[1]

$$P(Y > 2000) = \left(\frac{256.6038}{2000 + 256.6038} \right)^{2.6038} = 0.00348$$

[1]

(iii) Not separating out small and large claims results in an underestimation of tail probabilities for the claim distribution. This could be dangerous for estimating premium rates, reinsurance rates and security.

[2]

[Total 11]

Solution 12.

(i) Expected loss without reinsurance = *ë/(á 1)*= Rs. 1,500,000

[2]

Expected amount ceded to reinsurer is

$$\int_{2{,}000{,}000}^{\infty} x \frac{3.125\, l^{3.125}}{(l+x)^{4.125}}\, dx - 2{,}000{,}000 \left(\frac{l}{l + 2{,}000{,}000} \right)^a$$

$$= \left[-x \left(\frac{l}{l+x} \right)^{3.125} \right]_{2{,}000{,}000}^{\infty} + \int_{2{,}000{,}000}^{\infty} \frac{l^{3.125}}{(l+x)^{3.125}}\, dx - 2{,}000{,}000 \left(\frac{l}{l + 2{,}000{,}000} \right)^a$$

$$= 2{,}000{,}000 \left(\frac{l}{l + 2{,}000{,}000} \right)^a + \left[- \frac{l^{3.125}}{2.125(l+x)^{2.125}} \right]_{2{,}000{,}000}^{\infty} - 2{,}000{,}000 \left(\frac{l}{l + 2{,}000{,}000} \right)^a$$

$$= \left[-\frac{l^{3.125}}{2.125(l+x)^{2.125}} \right]_{2,000,000}^{\infty}$$

$$= \frac{3{,}187{,}500^{3.125}}{2.125\left(5{,}187{,}500\right)^{2.125}} = 532{,}889$$

Therefore the amount ceded to reinsurer is Rs532,889. Expected amount payable with reinsurance = Rs1,500,000 – Rs532,889 = Rs967,111

[3]

(ii) Annual expected profits under different scenarios

D_0 Reinsurance premium Rs0
Net total claims: $\grave{e}_0$ = Rs0
$\grave{e}_1$ = Rs1,500,000
$\grave{e}_2$ = Rs1,500,000×2= Rs3,000,000
$\grave{e}_3$ = Rs1,500,000×3= Rs4,500,000

D_1 Reinsurance premium Rs500,000
Net total claims: $\grave{e}_0$ = Rs0
$\grave{e}_1$ = Rs967,111
$\grave{e}_2$ = Rs967,111 + Rs1,500,000 = Rs2,467,111
$\grave{e}_3$ = Rs967,111 + Rs1,500,000×2= Rs3,967,111

D_2 Reinsurance premium Rs1,000,000
Net total claims: $\grave{e}_0$ = Rs0
$\grave{e}_1$ = Rs967,111
$\grave{e}_2$ = Rs967,111×2 = Rs1,934,222
$\grave{e}_3$ = Rs967,111×2 + Rs1,500,000= Rs3,434,222

The decision matrix is (figures in units of Rs1,000)

	No disaster $\grave{e}_0$	*1 disaster* $\grave{e}_1$	*2 disaster* $\grave{e}_2$	*3 or more disasters* $\grave{e}_3$
D_0	0	1,500	3,000	4,500
D_1	500	1,467	2,967	4,467
D_2	1,000	1,967	2,934	4,434

[6]

(iii) (a) Maximum losses:
D_0: Rs. 4,500
D_1: Rs. 4,467
D_2: Rs. 4,434 – minimum.

The minimax decision is D2.

[1]

(b) The insurer would expect to minimise the maximum loss by taking out the policy offering the most reinsurance, i.e. D_2.

[1]

(iv) P(0 claim) = exp(0.90) = 0.407,
P(1 claim) = 0.90*exp(*   0.90) = 0.365,
P(2 claim) = 0.90^2exp(-0.90)/2 = 0.165
P(> 2 claims) = 1   0.407   0.365   0.165   = 0.063.

[2]

Expected loss (in units of Rs1,000):

D_0: (0.407 × 0) + (0.365 × 1,500)+(0.165 × 3,000)+(0.063 × 4,500)=1,326 **minimum.**

D_1: (0.407 × 500) +(0.365 × 1,467)+(0.165 × 2,967)+(0.063 × 4,467)=1,510

D_2: (0.407 × 1, 000)+(0.365 × 1, 967)+(0.165 × 2, 934)+(0.063 × 4, 434)=1,888

Therefore, the answer is D_0: no reinsurance.

[2]

[Total 17]