Actuarial Society of India

Examinations

May 2006

CT6 – STATISTICAL MODELS

Indicative Solutions

1. (i) The pay-off matrix depicting losses for A is

		Player A		
		$x=1$	$x=2$	$x=3$
	$y=1$	6	3	2
Player B	$y=2$	3	6	4
	$y=3$	2	4	6

[2]

(ii) For player A,

$$\text{minimum loss} = \begin{cases} 2 & \text{if he chooses } x = 1, \\ 3 & \text{if he chooses } x = 2, \\ 2 & \text{if he chooses } x = 3. \end{cases}$$

Thus, maximin strategy is to choose $x = 2$. [1]

Also for player A, maximum loss is 6 for all three strategies, so all three choices of x are minimax. [1]

For player B, minimum loss is –6 for all three strategies, so all three choices of y are maximin. [1]

Also for player B,

$$\text{maximum loss} = \begin{cases} -2 & \text{if she chooses } y = 1, \\ -3 & \text{if she chooses } y = 2, \\ -2 & \text{if she chooses } y = 3. \end{cases}$$

Thus, minimax strategy is to choose $y = 2$. [1]

2. The original claim amount X has the Pareto distribution. Let the parameters of this distribution be α and λ. Then, the claim amount Z covered by the reinsurer, in respect of claims involving the reinsurer, has Pareto distribution with parameters α and $\lambda + 10,000$. We have

$$\begin{aligned} E(Z) &= (\lambda + 10,000)/(\alpha - 1); \\ E(Z^2) &= 2(\lambda + 10,000)^2/[(\alpha - 1)(\alpha - 2)]. \end{aligned}$$

[1]

The first two sample moments computed from the given data are 25,004.8 and 1,571,081,735. [2]

Solving the equations

$$\begin{aligned} (\lambda + 10,000)/(\alpha - 1) &= 25,004.8, \\ 2(\lambda + 10,000)^2/[(\alpha - 1)(\alpha - 2)] &= 1,571,081,735, \end{aligned}$$

we have the method of moments estimates $\hat{\alpha} = 5.90042$, $\hat{\lambda} + 10,000 = 122533.9$. Thus, $\hat{\lambda} = 112533.9$. [1]

The probability that a claim payment is shared by the reinsurer is $P(X > 10,000) = [\lambda/(\lambda + 10,000)]^{\alpha}$. Substituting the estimates of α and λ, we have the estimated proportion 0.3949. [1]

3. (i) Let the annual number of claims for a patient be N.

$$\begin{aligned} E(N) &= E(E(N|\theta)) = E(\lambda\theta) = \lambda\mu. \\ Var(N) &= E(Var(N|\theta)) + Var(E(N|\theta)) = E(\lambda\theta) + Var(\lambda\theta) \\ &= \lambda\mu + \lambda^2\mu^2 > E(N). \end{aligned}$$

[2]

(ii) $Var(N|\theta) = \lambda\theta = E(N|\theta)$. Thus, the conditional variance is the *same* as the conditional mean. The unconditional distribution of N is more dispersed (spread out) in relation to its mean – because of the additional uncertainty over income. [2]

(iii) We have $\lambda\mu + \lambda^2\mu^2 = 20$, which implies that $\lambda\mu = 4$. Since $\mu = 16,000$, $\lambda = 4/16000 = .00025$. [1]

(iv) Let X be a typical claim size and S be the total annual claim size.

$$\begin{aligned} E(S) &= E(N)E(X) = \lambda\mu\delta \\ Var(S) &= E(N)var(X) + var(N)[E(X)]^2 \\ &= \lambda\mu(\delta^2) + (\lambda\mu + \lambda^2\mu^2)\delta^2 = \lambda\mu\delta^2(2 + \lambda\mu) \end{aligned}$$

[2]

(v) Initially, condition everything on θ.

$$\begin{aligned} E(S|\theta) &= E(N|\theta)E(X|\theta) = (\lambda\theta)(\alpha\theta) = \alpha\lambda\theta^2, \\ var(S|\theta) &= E(N|\theta)var(X|\theta) + var(N|\theta)[E(X|\theta)]^2 \\ &= (\lambda\theta)(\alpha\theta)^2 + (\lambda\theta)(\alpha\theta)^2 = 2\alpha^2\lambda\theta^3. \end{aligned}$$

Now we can use the distribution of θ to calculate the unconditional mean and variance of S.

$$\begin{aligned} E(S) &= E(E(S|\theta)) = E(\alpha\lambda\theta^2) = 2\alpha\lambda\mu^2, \\ Var(S) &= E(var(S|\theta)) + var(E(S|\theta)) \\ &= E(2\alpha^2\lambda\theta^3) + var(\alpha\lambda\theta^2) \\ &= 2\alpha^2\lambda E(\theta^3) + \alpha^2\lambda^2[E(\theta^4) - \{E(\theta^2)\}^2] \\ &= 12\alpha^2\lambda\mu^3 + 20\alpha^2\lambda^2\mu^4. \end{aligned}$$

[3]

4. Let $X_1, X_2, \ldots, X_{100}$ be the claim sizes. We have for $i = 1, 2, \ldots, 10$,

$$\begin{aligned} E(X_i) &= e^{\mu+\sigma^2/2} = e^{10.02} = 22471.4, \\ var(X_i) &= e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) = 4539.6^2. \end{aligned}$$

[1]

Let $I_1, I_2, \ldots, I_{100}$ be the indicators of claim. Then, for the total claim amount $S = \sum_{i=1}^{100} I_i X_i$,

$$\begin{aligned}
E(S) &= 100E(I_1)E(X_1) = 100 \cdot 0.05 \cdot 22471.4 = 112357.1, \\
Var(S) &= 100[E(I_1^2 X_1^2) - E\{(I_1 X_1)^2\}] \\
&= 100[0.05(4539.6^2 + 22471.4^2) - 1123.571^2] \\
&= 50016.2^2
\end{aligned}$$

[2]

If the per-head premium is P, the probability that claims do not exceed premium is

$$P[S \le 100P] = P[(S - 112357.1)/50016.2 \le (100P - 112357.1)/50016.2].$$

If the normal approximation for S is used, this probability is equal to 0.95 when $(100P - 112357.1)/50016.2 = 1.645$. Solving for P, we have $P = 1946.3$.

The premium loading ξ satisfies the equation $100P = (1 + \xi)E(S)$. Solving it, we have $\xi = 0.7323$. [2]

5. (i) Let the mean number of claims for the ith year be μ_i. The model is

$$P(N_i = y) = \frac{e^{-\mu_i} \mu_i^y}{y!} = e^{y \log \mu_i - \mu_i - \log(y!)}, \quad i = 1, \ldots, n,$$

where

$$g(\mu_i) = \beta_0 + \beta_1 x_i, \quad i = 1, \ldots, n.$$ [2]

The log-likelihood is

$$\begin{aligned}
\ell &= \sum_{i=1}^{n} [N_i \log \mu_i - \mu_i - \log(N_i!)] \\
&= \sum_{i=1}^{n} [N_i \log\{g^{-1}(\beta_0 + \beta_1 x_i)\} - g^{-1}(\beta_0 + \beta_1 x_i) - \log(N_i!)] \\
&= \sum_{i=1}^{m} [N_i \log\{g^{-1}(\beta_0)\} - g^{-1}(\beta_0) - \log(N_i!)] \\
&\quad + \sum_{i=m+1}^{n} [N_i \log\{g^{-1}(\beta_0 + \beta_1)\} - g^{-1}(\beta_0 + \beta_1) - \log(N_i!)] \\
&= \log\{g^{-1}(\beta_0)\} \sum_{i=1}^{m} N_i - g^{-1}(\beta_0) m + \log\{g^{-1}(\beta_0 + \beta_1)\} \sum_{i=m+1}^{n} N_i \\
&\quad - g^{-1}(\beta_0 + \beta_1)(n - m) - \sum_{i=1}^{n} \log(N_i!).
\end{aligned}$$

[2]

(ii) The likelihood can be written as

$$\ell = \log a \sum_{i=1}^{m} N_i - am + \log b \sum_{i=m+1}^{n} N_i - b(n-m) + \text{constant},$$

where, $a = g^{-1}(\beta_0)$ and $b = g^{-1}(\beta_0 + \beta_1)$. Differentiating ℓ with the respect to a and b and setting the derivatives equal to zero, we have

$$(1/a)\sum_{i=1}^{m} N_i - m = 0, \quad (1/b)\sum_{i=m+1}^{n} N_i - (n-m) = 0.$$

These equations lead to the unique solution

$$\hat{a} = \sum_{i=1}^{m} N_i/m, \quad \hat{b} = \sum_{i=m+1}^{n} N_i/(n-m). \qquad [2]$$

The second derivative (hessian) matrix is

$$\begin{pmatrix} \frac{\partial^2 \ell}{\partial a^2} & \frac{\partial^2 \ell}{\partial a \partial b} \\ \frac{\partial^2 \ell}{\partial b \partial a} & \frac{\partial^2 \ell}{\partial b^2} \end{pmatrix} = \begin{pmatrix} -(1/a^2)\sum_{i=1}^{m} N_i & 0 \\ 0 & -(1/b^2)\sum_{i=m+1}^{n} N_i \end{pmatrix},$$

which is evidently a diagonal matrix with negative diagonal elements. Thus, $\hat{a}$ and $\hat{b}$ indeed correspond to the unique maximum likelihood estimators. [1]

The corresponding MLE of β_0 and β_1 are:

$$\begin{aligned} \hat{\beta}_0 &= g\left(\sum_{i=1}^{m} N_i/m\right), \\ \hat{\beta}_1 &= g\left(\sum_{i=m+1}^{n} N_i/(n-m)\right) - g\left(\sum_{i=1}^{m} N_i/m\right). \end{aligned}$$

[1]

(iii) The fitted value of μ_i is

$$\begin{aligned} \hat{\mu}_i &= g^{-1}(\hat{\beta}_0 + \hat{\beta}_1 x_i) \\ &= \begin{cases} \hat{a} & \text{if } 1 \le i \le m, \\ \hat{b} & \text{if } m < i \le n. \end{cases} \\ &= \begin{cases} \sum_{i=1}^{m} N_i/m & \text{if } 1 \le i \le m, \\ \sum_{i=m+1}^{n} N_i/(n-m) & \text{if } m < i \le n. \end{cases} \end{aligned}$$

These fitted values do not depend on g. [2]

(iv) No. The choice of g did not matter because its value at only two possible values of x_i were needed, and there are two parameters (β_0 and β_1) to adjust. This will not work when x_i can have more that two values. [2]

(v) The canonical link function is $g(\mu) = \log(\mu)$, as is evident from the first equation of part (i). [1]

(vi) The scaled deviance under the model is $2(\ell_S - \ell_M)$, where ℓ_S is the log-likelihood for the saturated model (where N_i itself is the estimator of μ_i), and

$$\begin{aligned}\ell_M &= \sum_{i=1}^{n}[N_i \log \hat{\mu}_i - \hat{\mu}_i - \log(N_i!)] \\ &= \sum_{i=1}^{m}[N_i \log \hat{a} - \hat{a} - \log(N_i!)] + \sum_{i=m+1}^{n}[N_i \log \hat{b} - \hat{b} - \log(N_i!)],\end{aligned}$$

where $\hat{a} = \sum_{i=1}^{m} N_i/m$ and $\hat{b} = \sum_{i=m+1}^{n} N_i/(n-m)$. Thus, the scaled deviance is

$$2\sum_{i=1}^{n}[N_i \log N_i - N_i - \log(N_i!)]$$

$$-2\sum_{i=1}^{m}[N_i \log \hat{a} - \hat{a} - \log(N_i!)] - 2\sum_{i=m+1}^{n}[N_i \log \hat{b} - \hat{b} - \log(N_i!)]. \quad [2]$$

(vii) For the model under constraint $\beta_1 = 0$, it can be easily verified that the MLE for the common value of the μ_is is $\sum_{i=1}^{n} N_i/n$. Let us denote this expression by $\hat{c}$. The corresponding log-likelihood is

$$\ell_{M_0} = \sum_{i=1}^{n}[N_i \log \hat{c} - \hat{c} - \log(N_i!)] .$$

The given expression for scaled deviance, $2(\ell_S - \ell_{M_0})$, follows easily. [2]

(viii) The hypothesis to be tested is $\beta_1 = 0$, or $b = a$.

This hypothesis can be tested by means of the change in scaled deviance as one switches from the model with $\beta_1 = 0$ to the model without this constraint. [1]

It follows from parts (vi) and (vii) that

$$\begin{aligned}&2(\ell_S - \ell_M) - 2(\ell_S - \ell_{M_0}) \\ &= 2(\ell_{M_0} - \ell_M) \\ &= 2\sum_{i=1}^{n}[N_i \log \hat{c} - \hat{c}] - 2\sum_{i=1}^{m}[N_i \log \hat{a} - \hat{a}] - 2\sum_{i=m+1}^{n}\left[N_i \log \hat{b} - \hat{b}\right] \\ &= 2\sum_{i=1}^{m} N_i \log(\hat{c}/\hat{a}) + 2\sum_{i=m+1}^{n} N_i \log(\hat{c}/\hat{b}) \\ &\quad -2m(\hat{c} - \hat{a}) - 2(n-m)(\hat{c} - \hat{b}),\end{aligned}$$

with

$$\hat{a} = \sum_{i=1}^{m} N_i/m, \quad \hat{b} = \sum_{i=m+1}^{n} N_i/(n-m), \quad \hat{c} = \sum_{i=1}^{n} N_i/n. \quad [1]$$

The asymptotic distribution of $2(\ell_{M_0} - \ell_M)$ is χ^2 with one degree of freedom, which can be used to obtain the p-value. [1]

6. (i) The characteristic equation is

$$1 - z - .5z^2 + .5z^3 = 0.$$

The cubic polynomial of the left hand side factorizes as $(1-z)(1-.5z^2)$. There is exactly one root on the unit circle. Therefore, $d = 1$. [1]

Rewriting the model in terms of $X = (1-B)Y$, we have

$$X_t - .5X_{t-2} = Z_t + .3Z_{t-1},$$

which is ARMA(2,1). Thus, the model for Y_t is ARIMA(2,1,1).[1]

(ii) The characteristic polynomial of X is $(1 - .5z^2)$, whose roots are $\pm\sqrt{2}$. As the roots are outside the unit circle, the process $\{X_t\}$ is stationary. [2]

(iii) The model equation is $X_t = .5X_{t-2} + Z_t + .3Z_{t-1}$. By taking covariances of both sides of this equation with Z_t, Z_{t-1} and Z_{t-2}, we have

$$\begin{aligned}
cov(X_t, Z_t) &= cov(.5X_{t-2} + Z_t + .3Z_{t-1}, Z_t) \\
&= 0 + \sigma^2 + 0 = \sigma^2, \\
cov(X_t, Z_{t-1}) &= cov(.5X_{t-2} + Z_t + .3Z_{t-1}, Z_{t-1}) \\
&= 0 + 0 + .3\sigma^2 = .3\sigma^2, \\
cov(X_t, Z_{t-2}) &= cov(.5X_{t-2} + Z_t + .3Z_{t-1}, Z_{t-2}) \\
&= .5\sigma^2 + 0 + 0 = .5\sigma^2.
\end{aligned}$$

[2]

By taking covariances of both sides of the model equation with X_t, X_{t-1}, X_{t-2} and X_{t-k} (for $k > 2$), we have

$$\begin{aligned}
\gamma(0) &= cov(X_t, X_t) = cov(.5X_{t-2} + Z_t + .3Z_{t-1}, X_t) \\
&= .5\gamma(2) + \sigma^2 + .09\sigma^2 = .5\gamma(2) + 1.09\sigma^2, \qquad (1) \\
\gamma(1) &= cov(X_t, X_{t-1}) = cov(.5X_{t-2} + Z_t + .3Z_{t-1}, X_{t-1}) \\
&= .5\gamma(1) + 0 + .3\sigma^2 = .5\gamma(1) + .3\sigma^2, \qquad (2) \\
\gamma(2) &= cov(X_t, X_{t-2}) = cov(.5X_{t-2} + Z_t + .3Z_{t-1}, X_{t-2}) \\
&= .5\gamma(0) + 0 + 0 = .5\gamma(0), \qquad (3) \\
\gamma(k) &= cov(X_t, X_{t-k}) = cov(.5X_{t-2} + Z_t + .3Z_{t-1}, X_{t-k}) \\
&= .5\gamma(k-2) + 0 + 0 = .5\gamma(k-2), \quad k > 2. \qquad (4)
\end{aligned}$$

[2]

By substituting for $\gamma(2)$ from (3) into (1), we have $\gamma(0) = .25\gamma(0) + 1.09\sigma^2$, i.e., $\gamma(0) = 109\sigma^2/75$. Equation (2) implies $\gamma(1) = 3\sigma^2/5$. Thus, $\rho(1) = \gamma(1)/\gamma(0) = 45/109$. Equations (3) and (4) together imply $\rho(k) = .5\rho(k-2)$ for $k \geq 2$. It follows that

$$\rho(k) = \begin{cases} (.5)^{|k|/2} & \text{if } |k| \text{ is even,} \\ (45/109)(.5)^{(|k|-1)/2} & \text{if } |k| \text{ is odd.} \end{cases} \qquad [2]$$

7. (i) Let the prior distribution be Beta(α, β). Prior density is

$$f(q) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} q^{\alpha-1}(1-q)^{\beta-1}, \quad 0 < q < 1.$$

Hence, the posterior density is proportional to

$$q^5(1-q)^{245} q^{\alpha-1}(1-q)^{\beta-1}, \quad 0 < q < 1.$$

Therefore, the posterior distribution is Beta with parameters $\alpha+5$ and $\beta+245$. [2]

Given the mean and variance of the prior distribution, we have

$$\frac{\alpha}{\alpha+\beta} = .015, \qquad \frac{\alpha}{(\alpha+\beta)^2} \cdot \frac{\beta}{(\alpha+\beta+1)} = .005^2. \qquad [2]$$

It follows from the mean equation that $\beta = 197\alpha/3$. Substituting this value in the variance equation, we get

$$\frac{.015^2}{\alpha} \cdot \frac{197\alpha/3}{(200\alpha/3+1)} = .005^2.$$

Eventually, we get $\alpha = 8.85$, $\beta = 581.15$.

The posterior distribution is Beta(13.85,826.15). [2]

(ii) The Bayes estimator under the squared error loss function is the posterior mean,

$$\frac{\alpha+5}{\alpha+5+\beta+245} = \frac{13.85}{840} = 0.0165. \qquad [2]$$

(iii) The Bayes estimator under the all-or-nothing loss function is the posterior mode, which is the solution of

$$(\alpha+5-1)x^{(\alpha+5-2)}(1-x)^{(\beta+245-1)} - x^{(\alpha+5-1)}(\beta+245-1)(1-x)^{(\beta+245-2)} = 0.$$

[1]

Therefore, the solution is

$$x = \frac{\alpha+5-1}{\alpha+5+\beta+245-2} = \frac{12.85}{838} = 0.0153. \qquad [1]$$

8. (i)

$$\begin{aligned} E(\alpha) &= E\left(e^{\mu+\sigma^2/2}\right) \\ &= e^{\sigma^2/2}\int_{-\infty}^{\infty} e^{\mu}(2\pi\tau^2)^{-1/2}e^{-(\mu-\theta)^2/2\tau^2}d\mu \\ &= e^{\theta+\sigma^2/2}\int_{-\infty}^{\infty} e^{(\mu-\theta)}(2\pi\tau^2)^{-1/2}e^{-(\mu-\theta)^2/2\tau^2}d\mu \\ &= e^{\theta+\sigma^2/2}\int_{-\infty}^{\infty} e^{u\tau}(2\pi)^{-1/2}e^{-u^2/2}du \\ &= e^{\theta+\sigma^2/2}\int_{-\infty}^{\infty} (2\pi)^{-1/2}e^{-(u^2-2u\tau)/2}du \\ &= e^{\theta+\sigma^2/2+\tau^2/2}\int_{-\infty}^{\infty} (2\pi)^{-1/2}e^{-(u-\tau)^2/2}du \\ &= e^{\theta+\sigma^2/2+\tau^2/2}. \end{aligned}$$

[3]

(ii) Let $Y_i = \log X_i$, $i = 1, 2, \ldots, n$ and $\bar{Y} = n^{-1}\sum_{i=1}^{n} Y_i$. Using the normal-normal model, the posterior distribution of μ is seen to be normal with mean $(n\bar{Y}/\sigma^2 + \theta/\tau^2)/(n/\sigma^2 + 1/\tau^2)$ and variance $(n/\sigma^2 + 1/\tau^2)^{-1}$. Thus, the posterior mean of α can be obtained by replacing θ and τ^2 in the expression of the prior mean of α, by $z\bar{Y} + (1-z)\theta$ and $(n/\sigma^2 + 1/\tau^2)^{-1}$. The expression given in the question follows. [2]

9. In one year

$P[0 \text{ claim}]$	$= e^{-0.2}$	$= 0.8187,$
$P[1 \text{ claim}]$	$= 0.2e^{-0.2}$	$= 0.1637,$
$P[2 \text{ claims}]$	$= 0.2^2e^{-0.2}/2$	$= 0.0164,$
$P[\geq 3 \text{ claims}]$	=1 – sum of above	$= 0.0012.$

[2]

Transition matrix is Π so that $x_nP = x_{n+1}$,
where $x_1 = (0, 0, 0, 0, 0, 10000, 0)$.

$$\Pi = \{\pi_{ij}\}, \quad \pi_{ij} = P[\text{Class } j \text{ next year} \mid \text{Class } i \text{ this year}].$$

$$\Pi = \begin{pmatrix} 0.8187 & 0 & 0 & 0.1637 & 0 & 0.0164 & 0.0012 \\ 0.8187 & 0 & 0 & 0.1637 & 0 & 0.0164 & 0.0012 \\ 0 & 0.8187 & 0 & 0 & 0.1637 & 0 & 0.0176 \\ 0 & 0 & 0.8187 & 0 & 0.1637 & 0 & 0.0176 \\ 0 & 0 & 0 & 0.8187 & 0 & 0.1637 & 0.0176 \\ 0 & 0 & 0 & 0 & 0.8187 & 0 & 0.1813 \\ 0 & 0 & 0 & 0 & 0 & 0.8187 & 0.1813 \end{pmatrix}.$$

[4]

$$\begin{aligned} x_2 = x_1P &= (0, 0, 0, 0, 8187, 0, 1813). \\ x_3 = x_2P &= (0, 0, 0, 6702.7, 0, 1340.2 + 1484.3, 144.1 + 328.7) \\ &= (0, 0, 0, 6702.7, 0, 2824.5, 472.8). \end{aligned}$$

[2]

10. Assumptions :

A loss ratio developed from years 1997-2000 is a reasonable a-priori estimate for years 2001-2005.
There are no outstanding claims for pre-2001 years.
The chain ladder method and its assumptions are applicable [2]

Acc. Year	Development Year						Ult	Earned Premium	Est. LR
	0	**1**	**2**	**3**	**4**	**5**			
1997	2,323	2,713	2,902	3,009	3,081	3,065	3,065	3,606	85.00%
1998	2,489	2,907	3,109	3,224	3,301	3,287	3,287	3,864	85.07%
1999	2,709	3,165	3,385	3,509	3,393	3,572	3,572	4,206	84.93%
2000	2,966	3,464	3,705	3,842	3,934	3,914	3,914	4,604	85.01%
									Average 85.00%
2001	3,512	4,042	4,205	4,394	4,458			5,305	
2002	4,054	4,610	4,938	5,101				5,896	
2003	4,614	5,421	5,690					6,578	
2004	5,354	6,180						7,546	
2005	5,700							8,304	
TOTAL	33,721	32,502	27,934	23,079	18,167	13,838	13,838		
(1997-2005)									
Tot-last	28,021	26,322	22,244	17,978	13,709				
Dev. F	1.160	1.061	1.038	1.011	1.009	1.000			
Cum. F	1.304	1.124	1.059	1.020	1.009	1.000			

[4]

	Accident Year					
	2005	**2004**	**2003**	**2002**	**2001**	**2000**
Est. Ult. Cl. LR 85%	7,058	6,414	5,591	5,012	4,509	3,913
Exp Inc.	5,413	5,707	5,280	4,914	4,469	3,913
Emg Res.	1,645	707	311	98	40	0
Inc. Cl	5,700	6,180	5,690	5,101	4,458	3,914
Ultimate Liab.	7,345	6,887	6,001	5,199	4,498	3,914

[4]

Overall totals Ultimate Liab. Years 2001-05 29,930
Paid Claims <u>20,485</u>

Reserve for Outstanding & IBNR 9,445 [2]

11. Note that $X_1 + \cdots + X_n$ has the gamma(n, μ) distribution. Therefore,

$$\begin{aligned}
P(N = n) &= P(X_1 + \cdots + X_n \leq t \leq X_1 + \cdots + X_{n+1}) \\
&= \int_0^t P(X_1 + \cdots + X_n \leq t \leq X_1 + \cdots + X_{n+1} | X_1 + \cdots + X_n = x) \\
&\quad \times \frac{\mu^n x^{n-1}}{(n-1)!} \cdot e^{-x/\mu} dx \\
&= \int_0^t P(X_{n+1} \geq t - x) \frac{x^{n-1}}{\mu^n (n-1)!} \cdot e^{-x/\mu} dx \\
&= \int_0^t e^{-(t-x)/\mu} \frac{x^{n-1}}{\mu^n (n-1)!} \cdot e^{-x/\mu} dx \\
&= \int_0^t \frac{x^{n-1}}{\mu^n (n-1)!} \cdot e^{-t/\mu} dx \\
&= \frac{1}{\mu^n (n-1)!} \cdot e^{-t/\mu} \int_0^t x^{n-1} dx \\
&= \frac{1}{\mu^n (n-1)!} \cdot e^{-t/\mu} \frac{t^n}{n} \\
&= \frac{e^{-t/\mu}}{n!} (t/\mu)^n.
\end{aligned}$$

This is clearly the Poisson probability function with mean t/μ. [3]

In order to generate a sample from the Poisson distribution with mean λ, generate independent uniformly distributed (over 0 to 1) random numbers $U_1, U_2, \ldots$, and let $X_i = -\log(U_i)/\lambda$, for $i = 1, 2, \ldots$. Then the X_i's are iid exponential with mean $1/\lambda$. Define N as the largest number such that the sum $X_1 + \cdots + X_N$ does not exceed 1. Then N has the requisite Poisson distribution. This follows from the above result with $\mu = 1/\lambda$ and $t = 1$. [3]

12. $(1 - \alpha B)Y_t = Z_t$

$$Y_t = 1 / (1 - \alpha B) * Z_t$$
$$= (1 + \alpha B + \alpha^2 B^2 + \ldots.) * Z_t$$
$$= Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \ldots\ldots$$ [1]

$$V(Y_t) = (1 + \alpha^2 + \alpha^4 + \alpha^6 + \ldots.)\ \sigma^2$$
$$= 1 / (1 - \alpha^2) * \sigma^2$$ [2]