ENGINEERING & MANAGEMENT EXAMINATIONS, DECEMBER – 2008

MATHEMATICS

SEMESTER - 1

Time : 3 Hours] [Full Marks : 70

GROUP – A

(Multiple Choice Type Questions)

1. Choose the correct alternatives for any *ten* of the following : $10 \times 1 = 10$

i) The value of $\lim_{x \to 0} \frac{\log \sin x}{6\, tx}$ is

a) 0 b) $\frac{1}{2}$

c) 1 d) none of these.

ii) The sequence $\left\{ \frac{1}{3^n} \right\}$ is

a) monotonic increasing b) oscillatory

c) divergent d) monotonic decreasing.

iii) The distance between the two parallel planes $x + 2y - z = 4$ and $2x + 4y - 3z = 3$ is

a) $\frac{5}{\sqrt{24}}$ b) $\frac{5}{24}$

c) $\frac{11}{\sqrt{24}}$ d) none of these.

iv) n^{th} derivative of $\sin(5x + 3)$ is

a) $5^n.\cos(5x + 3)$ b) $5^n.\sin\left(\frac{n\pi}{2} + 5x + 3\right)$

c) $15.\sin\left(\frac{n\pi}{2} + 5x + 3\right)$ d) $-\sin(5x + 3)$.

v) If $u(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$, then the value of $x\frac{\partial u}{\partial x} + y\frac{\partial}{\partial y}$ is

a) 0 b) $2u(x, y)$

c) $u(x, y)$ d) none of these.

vi) If $f(x)$ is continuous in $[a, a+h]$, derivable in $(a, a+h)$ then $f(a+h) - f(a) = hf(a+\theta h)$, where

a) θ is any real b) $0 < \theta < 1$

c) $\theta > 1$ d) θ is an integer.

vii) The value of $\int_1^0 \int_0^1 (x+y)\,dxdy =$

a) 2 b) 3

c) 1 d) 0.

viii) The series $\sum \frac{1}{n^p}$ is convergent if

a) $p \geq 1$ b) $p > 1$

c) $p < 1$ d) $p \leq 1$.

ix) Value of $\int_C x\,dy$ where C is the arc cut off from the parabola $y^2 = x$ from the point $(0, 0)$ to $(1, -1)$ is

a) $-\frac{1}{3}$ b) $\frac{1}{3}$

c) 0 d) none of these.

x) $\int_0^{\pi/2} \sin^2 x \, dx =$

a) $\frac{7}{15}$ b) $\frac{8}{15}$

c) $\frac{8\pi}{15}$ d) $\frac{4}{15}$.

xi) If $u + v = x$, $uv = y$, then $\frac{\partial(x, y)}{\partial(u, v)} =$

a) $u - v$ b) uv

c) $u + v$ d) u/v.

xii) If $f(x) = \frac{1 - \sin x}{\sin 2x}$, $x \neq \frac{\pi}{2}$ is continuous at $x = \frac{\pi}{2}$ then $f\left(\frac{\pi}{2}\right) =$

a) $\frac{1}{2}$ b) 1

c) -1 d) 0.

xiii) The value of the constant p, so that the vector function $\vec{f} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + pz)\hat{k}$ is solenoidal, is

a) -1 b) 2

c) -2 d) 1.

xiv) If $\vec{\alpha} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{\beta} = 2\hat{i} - \hat{k}$, then $(\vec{\alpha} \times \vec{\beta}) \cdot \vec{\alpha}$ is equal to

a) 0 b) 1

c) $\frac{1}{2}$ d) -1.

xv) The limit $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2 + y^2}$ does not exist.

a) True b) False.

GROUP – B

(Short Answer Type Questions)

Answer any *three* of the following. $3 \times 5 = 15$

2. Prove that if, $I_n = \int_0^{\pi/2} x^n \sin x \, dx$, then $I_n + n(n-1) I_{n-2} = n(\pi/2)^{n-1}$.

3. Test the convergence of the series

$$\sum_{n=1}^{\infty} \left(\sqrt{n^4+1} - \sqrt{n^4-1} \right).$$

4. If $f(x) = \sin^{-1} x$, $0 < a < b < 1$, use mean value theorem to prove

$$\frac{b-a}{\sqrt{(1-a^2)}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{(1-b^2)}}.$$

5. Show that $\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n \frac{n!}{x^{n-1}} \left(\log x - 1 - 1/2 - 1/3 - \ldots\ldots - 1/n \right)$.

6. Find the values of a and b such that

$$\lim_{\theta \to 0} \frac{\theta(1 + a\cos\theta) - b\sin\theta}{\theta^3} = 1.$$

7. Find the equation of the sphere having the circle $x^2 + y^2 + z^2 - 10y + 4z - 8 = 0$, $x + y + z = 6$ as a great circle.

GROUP – C

(Long Answer Type Questions)

Answer any *three* of the following. $3 \times 15 = 45$

8. a) Using mean value theorem prove that

$$\frac{x}{1+x^2} < \tan^{-1} x < x, \quad 0 < x < \frac{\pi}{2}.$$ 5

b) If z is a function of x and y and $x = r\cos\theta$, $y = r\sin\theta$ then prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}.$$ 5

c) If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, $f(x) = 1/(1+x)$ and $h = 7$, find θ. 5

9. a) Show that for the function

$$f(x, y) = \begin{cases} \frac{x^2 y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$f_{xy}(0, 0) = f_{yx}(0, 0)$. 5

b) State comparison test for convergence of an infinite series. Test the convergence of any *one* of the following series :

i) $\frac{6}{1.3.5} + \frac{8}{4.5.7} + \frac{10}{5.7.9} + \ldots\ldots$

ii) $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots (p > 0)$. 5

c) Find the extreme values, if any, of the following function :

$$f(x, y) = x^3 + y^3 - 3axy.$$ 5

10. a) Obtain the reduction formula for $\int_0^{\pi/2} \cos^n x\, dx$. Hence evaluate $\int_0^{\pi/2} \cos^5 x\, dx$. 5

b) Compute the value of $\iint_R y\, dx dy$ where R is the region in the first quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. 5

c) Obtain the reduction formula for $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$, where m, n are positive integers $(m > 1, n > 1)$. Hence evaluate

$$\int_0^{\pi/2} \sin^4 x \cos^8 x\, dx.$$ 5

11. a) If $u = \sin^{-1}\sqrt{\dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}}}$ then verify whether the following indentity is ture :

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right).$$ 5

b) Find the angle between the surfaces $x^3 + y^3 + z^3 - 3xyz = 5$ and $x^2 y + y^2 z + z^2 x - 5xyz = 8$ at the point (1, 0, 1). 5

c) Evaluate $\left[\dot{\vec{r}} \;\; \ddot{\vec{r}} \;\; \dddot{\vec{r}} \right]$ where $\vec{r} = a \cos u \,\hat{i} + a \sin u \,\hat{j} + bu \,\hat{k}$. 5

12. a) A variable plane passes through a fixed point (a, b, c) and meets the coordinate axes at A, B, C. Show that the locus of the point of intersection of the plane through A, B, C and parallel to the coordinate planes is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$. 5

b) Show that the straight lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-5}{4} = \frac{y-4}{4} = \frac{z-5}{5}$ are coplanar. 5

c) Find the length of the perimeter of the asteroid, $x^{\frac{2}{3}} + x^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Determine also the length of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. $2 \times 2\frac{1}{2} = 5$

13. a) Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2 + 1}$ Is it absolutely convergent ? 5

b) Find the directional derivative of $f(x, y, z) = x^2 yz + 4xz^2$ at the point (1, 2, – 1) in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$. 5

c) Find the moment of inertia of a thin uniform lamina in the form of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major and minor axes respectively. 5

END