Test Code MS (Short answer type) 2006
Syllabus for Mathematics

Permutations and Combinations. Binomial and multinomial theorem. Theory of equations. Inequalities.

Determinants, matrices, solution of linear equations and vector spaces.

Trigonometry, Coordinate geometry of two and three dimensions.

Geometry of complex numbers and De Moivre's theorem. Elements of set theory.

Convergence of sequences and series. Power series. Functions, limits and continuity of functions of one or more variables.

Differentiation, Leibnitz formula, maxima and minima, Taylor's theorem. Differentiation of functions of several variables. Applications of differential calculus.

Indefinite integral, Fundamental theorem of calculus, Riemann integration and properties. Improper integrals. Differentiation under the integral sign. Double and multiple integrals and applications.

Syllabus for Statistics

Probability and Sampling Distributions

Notions of sample space and probability, combinatorial probability, conditional probability and independence, random variable and expectations, moments, standard discrete and continuous distributions, sampling distributions of statistics based on normal samples, central limit theorem, approximation of binomial to normal or Poisson law. Bivariate normal and multivariate normal distributions.

Descriptive Statistics

Descriptive statistical measures, graduation of frequency curves, productmoment, partial and multiple correlation, Regression (bivariate and multivariate).

Inference

Elementary theory and methods of estimation (unbiasedness, minimum variance, sufficiency, maximum likelihood method, method of moments). Tests of hypotheses (basic concepts and simple applications of Neyman-Pearson Lemma). Confidence intervals. Tests of regression. Elements of non-parametric inference.

Design of Experiments and Sample Surveys

Basic designs (CRD/RBD/LSD) and their analyses. Elements of factorial designs. Conventional sampling techniques (SRSWR/SRSWOR) including stratification; ratio and regression methods of estimation.

Sample Questions

1. Let A and B be two invertible $n \times n$ real matrices. Assume that $A+B$ is invertible. Show that $A^{-1} + B^{-1}$ is also invertible.

2. Maximize $x + y$ subject to the condition that $2x^2 + 3y^2 \leq 1$.

3. Let $X_1, X_2 \ldots$ be i.i.d. Bernoulli random variables with parameter $\frac{1}{4}$, let $Y_1, Y_2, \ldots$ be another sequence of i.i.d. Bernoulli random variables with parameter $\frac{3}{4}$ and Let N be a geometric random variable with parameter $\frac{1}{2}$ (i. e. $P(N = k) = \frac{1}{2^k}$ for $k = 1, 2 \ldots$). Assume the X_i's, Y_j's and N are all independent.

   Compute $\text{Cov}(\sum_{i=1}^{N} X_i, \sum_{i=1}^{N} Y_i)$.

4. Let U be uniformly distributed on the interval $(0, 2)$ and let V be an independent random variable which has a discrete uniform distribution on $\{0, 1, \ldots, n\}$. *i.e.*

$$P\{V = i\} = \frac{1}{n+1} \qquad \text{for } i = 0, 1, \ldots, n.$$

   Find the cumulative distribution function of $X = U + V$.

5. Suppose X is the number of heads in 10 tossses of a fair coin. Given $X = 5$, what is the probability that the first head occured in the third toss?

6. Let Y_1, Y_2, Y_3 be i.i.d. continuous random variables. For $i = 1, 2$, define U_i as

$$U_i = \begin{cases} 1 & \text{if } Y_{i+1} > Y_i, \\ 0 & \text{otherwise.} \end{cases}$$

   Find the mean and variance of $U_1 + U_2$.

7. Let Y be a random variable with probability density function

$$f_Y(y|\theta) = \begin{cases} \frac{1}{\theta} e^{-y/\theta} & \text{if } y > 0, \\ 0 & \text{otherwise,} \end{cases}$$

   with $\theta > 0$.

   Suppose that the conditional distribution of X given $Y = y$ is $N(y, \sigma^2)$, with $\sigma^2 > 0$. Both θ and σ^2 are unknown parameters. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from the joint distribution of X and Y.

   Find a nontrivial joint sufficient statistic for (θ, σ^2).

8. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from the discrete distribution with joint probability mass function

$$f_{X,Y}(x, y) = \begin{cases} \frac{\theta}{4} & (x, y) = (0, 0) \text{ and } (1, 1), \\ \frac{2-\theta}{4} & (x, y) = (0, 1) \text{ and } (1, 0), \end{cases}$$

with $0 \leq \theta \leq 2$. Find the maximum likelihood estimator of θ.

9. Let $X_1, X_2, \ldots$ be i.i.d. random variables with density $f_\theta(x)$, $x \in \mathbb{R}$, $\theta \in (0, 1)$ being the unknown parameter. Suppose that there exists an unbiased estimator T of θ based on sample size 1, i. e. $E_\theta(T(X_1)) = \theta$. Assume that $\text{Var}(T(X_1)) < \infty$.

   (a) Find an estimator V_n for θ based on $X_1, \ldots X_n$ such that V_n is consistent for θ.

   (b) Let S_n be the MVUE (minimum variance unbiased estimator) of θ based on $X_1, \ldots, X_n$. Show that $\lim_{n\to\infty} \text{Var}(S_n) = 0$.

10. For the data collected via a randomized block design with v treatments and b blocks, the following model is postulated:

$$E(y_{ij}) = \mu + \tau_i + \beta_j, \quad 1 \leq i \leq v, 1 \leq j \leq b,$$

where t_i and β_j are the effects of the ith treatment and the jth block respectively, and μ is a general mean. For $1 \leq i \leq v$, define $Q_i = T_i - \frac{G}{v}$, where T_i is the total of observations under the ith treatment and $G = \sum_{i=1}^{v} T_i$. Show that

$$E(Q_i) = (b - \frac{b}{v})\tau_i, \quad \text{Var}(Q_i) = \sigma^2(b - \frac{b}{v}),$$
$$\text{Cov}(Q_i, Q_j) = -(\frac{b}{v})\sigma^2 \text{ for } i \neq j,$$

where σ^2 is the per observation variance.

11. A straight line regression $E(y) = \alpha + \beta x$ is to be fitted using four observations. Assume $\text{Var}(y|x) = \sigma^2$ for all x. The values of x at which observations are to be made lie in the closed interval $[-1, 1]$. The following choices of the values of x where observations are to be made are available:

   (a) two observations each at $x = -1$ and $x = 1$,

   (b) one observation each at $x = -1$ and $x = 1$ and two observations at $x = 0$,

(c) one observation each at $x = -1, -\frac{1}{2}, \frac{1}{2}, 1$.

If the interest is to estimate the slope with least variance, which of the above strategies would you choose and why?

12. Consider a possibly unbalanced coin with probability of heads in each toss being p, where p is unknown. Let X be the number of tails before the first head occurs. Find the uniformly most powerful test of level α for testing $H_0 : p = \frac{1}{6}$ against $H_1 : p > \frac{1}{6}$.