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Your Roll No

7241 **J**

M.Sc./II

OPERATIONAL RESEARCH—Course XIII

(Mathematical Programming)

(Admissions of 2001 and onwards)

Time 3 *Hours* *Maximum Marks* 75

(*Write your Roll No on the top immediately on receipt of this question paper*)

Attempt any **Five** *questions*

All *questions carry equal marks.*

1 (a) Check the following function for convexity

$$f(x) = \frac{1}{4}x_1^4 - x_1^2 + x_2^2$$

(b) Let $S \subseteq R^n$ be an open convex set and $f: S \rightarrow R$ be differentiable Show that f is convex if and only if

$$f(x) - f(u) \geq (x - u)^t \nabla f(u)$$

for all $x, u \in S$. Give a geometrical interpretation of the above inequality

(c) For the problem

min $f(x)$

subject to $x \in S$

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where $S \subseteq R^n$ is a convex set and f is a convex function, show that the set of all optimal solutions is a convex set

2 (a) Using the variable transformation, associate a linear programming problem with the following linear fractional programming problem (LFPP)

$$\text{Min} \frac{c'x+\alpha}{d'x+\beta}$$

Subject to $Ax \geq b$

$x \geq 0$

Let $(y^*\ z^*)$ be an optimal solution of the linear programming problem If $z^* \neq 0$ then show that $x^* = \frac{y}{z}$ is an optimal solution of the LFPP and the objective functions values of the LFPP and the corresponding linear programming problem are equal

(b) Solve the following linear fractional programming problem by Simplex method

$$\text{Max} \quad f(x_1, x_2) = \frac{5x_1 + 3x_2}{5x_1 + 2x_2 + 1}$$

subject to

$$3x_1 + 5x_2 \leq 15$$

$$5x_1 + 2x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

3 (a) Consider the problem

Min $f(x)$

subject to

$$Ax = b$$

$$x \geq 0$$

where A is an $m \times n$ matrix with rank m, b is an $m \times 1$ vector and $f(x)$ is convex continuously differentiable function Show that a feasible point x^0 is a KKT point if and only if $\alpha = \beta = 0$, where

$$\alpha = \max \{- r_j, r_j \leq 0\}$$

$$\beta = \max \{x_j r_j, r_j \geq 0\}$$

and $r^t = \nabla_N f(x^0) - \nabla_B f(x^0)^t B^{-1} N$

(N denotes the indices of non-basic variables)

(b) Give a brief outline of the theory of Simplex method for solving the following indefinite quadratic programming problem

$$\max f(x) = (c^t x + \alpha)(d^t x + \beta)$$

subject to

$$Ax \leq b$$

$$x \geq 0$$

where c, d are $n \times 1$ vectors A is an $m \times n$ matrix, b is $m \times 1$ vector and α, β are scalars

4 (a) Define a pseudoconvex function Give examples of such functions Let $S \subseteq R^n$ be an open convex set

and $f : S \to R$ be a pseudoconvex function on S. Show that f is also a quasiconvex function on S.

(b) Define a quasiconcave function and give examples of such functions Is the sum of two quasiconcave functions always a quasiconcave function ? Justify

(c) Use KKT conditions to find the value of β for which (1, 2) is optimal to the problem

$$\max \quad f(x) = 2x_1 + \beta x_2$$

subject to

$$x_1^2 + x_2^2 \leq 5$$

$$x_1 - x_2 \leq 2$$

5 (a) Develop Karush-Kuhn-Tucker (KKT) necessary optimality conditions for a point x^0 to be a local optimal solution of the non-linear programming problem

$$\min f(x)$$

subject to

$$g_i(x) \leq 0 \ (i = 1, 2, \ldots, m)$$

where $x \in R^n$ and f, g_i $(i = 1, 2, \ldots, m)$ are defined and continuously differentiable functions Define all the sets used in the development process and state the appropriate constraint qualification used

(b) Consider the following non-linear programming problem (NLP)

$$\min \quad f(x) = (x_1 - 4)^2 + (x_2 - 6)^2$$

subject to

$$x_2 \geq x_1^2$$

$$x_1 \leq 4$$

(i) Write the KKT conditions

(ii) Do the KKT conditions hold at the point (2, 4) ?

(iii) Can we use KKT conditions to conclude that (2, 4) is optimal ? Justify

6 (a) For the non-linear programming problem

$$\max f(x)$$

subject to

$$g_i(x) \geq 0 \quad (i = 1, 2, \ldots, m),$$

Write the Wolfe dual State and prove the weak Duality Theorem and Strong Duality Theorem under suitable assumptions on the functions involved

(b) Show that the Lagrangian function for the problem

$$\max e^x$$

subject to

$$-\frac{1}{2} \leq x \leq 1$$

does not possess a saddle point

7 (a) For the linear fractional programming problem

$$\max f(x) = \frac{x}{dx}, \quad (dx > 0 \text{ for all feasible } x)$$

subject to

$$Ax = b$$

$$x \geq 0$$

where all basic solutions to $Ax = b$, $x \geq 0$ exist and are non-degenerate, write the fractional dual State and prove the Strong Duality Theorem

(b) For the non-linear programming problem

$$\max f(x)$$

subject to

$$g_i(x) \geq 0 \quad (i = 1, 2, \ldots, m)$$

State and prove the sufficiency of KKT optimality conditions under the assumption that f is pseudoconcave and g_i's $(i = 1, 2, \ldots, m)$ are quasiconcave functions